COMPARISON OF VOLUMES BY MEANS OF THE AREAS OF CENTRAL SECTIONS.

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Abstract. The Busemann-Petty problem asks whether origin-symmetric convex bodies in $\mathbb{R}^n$ with smaller areas of all central hyperplane sections necessarily have smaller n-dimensional volume. The solution was completed in the end of the 90's, and the answer is affirmative if $n \leq 4$ and negative if $n \geq 5$. Since the answer is negative in most dimensions, it is natural to ask what information about the volumes of central sections of two bodies does allow to compare the n-dimensional volumes of these bodies in all dimensions. In this article we give an answer to this question in terms of certain powers of the Laplace operator applied to the section function of the body.

1. Introduction

For an origin-symmetric convex body K in $\mathbb{R}^n$, we denote by

$$S_K(\xi) = \mathrm{vol}_{n-1}(K \cap \xi^\perp), \qquad \xi \in S^{n-1}$$

the section function of K. Here $\xi^\perp$ stands for the central hyperplane in $\mathbb{R}^n$ orthogonal to ξ, and vol_{n-1} is the $(n-1)$-dimensional volume. We extend S_K from the sphere to the whole $\mathbb{R}^n$ as a homogeneous function of degree -1.

The classical Minkowski's uniqueness theorem (see for example [Ga]) shows that an origin-symmetric convex body is uniquely determined by its section function, namely if K and L are two origin-symmetric convex bodies in $\mathbb{R}^n$ with $S_K(\xi) = S_L(\xi)$ for every $\xi \in S^{n-1}$ then $K = L$. This is true even for origin-symmetric starshaped bodies, and, moreover, one can reconstruct every such body out of its section function by using the inverse formula for the spherical Radon transform ([He]) or the Fourier transform formula (see [K1, Th.1]):

$$\widehat{S_K}(x) = \frac{(2\pi)^n}{\pi(n-1)} \|x\|_K^{-n+1}, \tag{1}$$

where $\|x\|_K = \min\{a \geq 0 : \ x \in aK\}$ is the Minkowski functional of K, and the Fourier transform $\widehat{S_K}$ is in the sense of distributions.

In view of the uniqueness result, it is quite surprizing that the corresponding comparison theorem fails in almost all dimensions. Suppose that K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ so that $S_K(\xi) \leq S_L(\xi)$ for every $\xi \in S^{n-1}$. Does it follow that $\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)$? This question is the matter of the Busemann-Petty problem, posed in [BP] in 1956 and solved in the end of the 90's. The answer is affirmative if $n \leq 4$ and negative if $n \geq 5$. The solution appeared as the result of work of many mathematicians (see [GKS] or [Zh] for historical details).

Since the answer is negative in most dimensions, it is natural to ask what does one need to know about the $(n-1)$-dimensional volumes of central sections of two bodies to be able to compare their n-dimensional volumes in all dimensions. In this article we suggest the following answer:

Theorem 1. *Let $n \geq 4$ be an even integer, K, L origin-symmetric $(n-4)$-smooth convex bodies in $\mathbb{R}^n$ so that, for every $\xi \in S^{n-1}$,*

$$(-1)^{(n-4)/2}\Delta^{(n-4)/2} S_K(\xi) \leq (-1)^{(n-4)/2}\Delta^{(n-4)/2} S_L(\xi), \tag{2}$$

where Δ is the Laplace operator on $\mathbb{R}^n$. Then $\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)$.

Putting $n = 4$ one can see that Theorem 1 represents a generalization of the affirmative part of the solution to the Busemann-Petty problem. Note that another generalization of the Busemann-Petty problem was given in [K2], where the condition (2) was replaced by an inequality for the derivatives of parallel sections functions at zero. Computing these derivatives involves non-central sections, so the result of [K2] does not answer the question of this paper. The result of Theorem 1 holds true for odd n, but in this case one has to consider fractional powers of the Laplace operator. For other generalizations of the Busemann-Petty problem and related open questions see [BZ], [K3], [K4], [RZ], [MP].

2. Proof of Theorem 1

As usual, we denote by $\mathcal{S}$ the space of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ with values in $\mathbb{C}$. We use notation and results from [GS]. By $\mathcal{S}'$ we denote the space of distributions over $\mathcal{S}$. The *Fourier transform* of a distribution f is defined by $\langle \hat{f}, \hat{\varphi}\rangle = (2\pi)^n \langle f, \varphi\rangle$ for every test function φ. A well-known connection between the Fourier transform and differentiation implies that, for any distribution f, $(\Delta f)^\wedge = -|\cdot|_2^2 \hat{f}$, where $|\cdot|_2$ stands for the Euclidean norm in $\mathbb{R}^n$.

A distribution f is called *positive definite* if, for every test function φ,

$$\left\langle f, \varphi * \overline{\varphi(-x)} \right\rangle \geq 0.$$

By L.Schwartz's generalization of Bochner's theorem, a distribution is positive definite if and only if its Fourier transform is a positive distribution (in the sense that $\langle \hat{f}, \varphi\rangle \geq 0$ for every non-negative test function φ; see, for example, [GV, p. 152]).

We need the following fact proved in [K4, Corollary 4]:

Lemma 1. *Let K be an origin-symmetric convex body in $\mathbb{R}^n$, $n \geq 4$. For every $0 \leq k \leq n-3$ and every $p \in (0,3]$, the function $|x|_2^{-k}\|x\|_K^{-n+k+p}$ represents a positive definite distribution on $\mathbb{R}^n$.*

A body K in $\mathbb{R}^n$ is called k-smooth, $0 \leq k \leq \infty$, if the restriction of $\|\cdot\|_K$ to the sphere S^{n-1} belongs to the space $C^{(k)}(S^{n-1})$ of continuously differentiable up to the order k functions. As shown in [K2, Lemma 5], if K is infinitely smooth then, for any $0 < p < n$, the Fourier transform of $\|\cdot\|_K^{-p}$ is a homogeneous of degree $-n+p$ function on $\mathbb{R}^n$, whose restriction to the sphere is infinitely differentiable. We use a version of Parseval's formula on the sphere proved in [K2, Lemma 3]:

Lemma 2. *Let K and L be origin-symmetric infinitely smooth convex bodies in $\mathbb{R}^n$ and $0 < p < n$. Then*

$$\int_{S^{n-1}} (\|x\|_K^{-p})^\wedge(\theta)(\|x\|_L^{-n+p})^\wedge(\theta)\, d\theta = (2\pi)^n \int_{S^{n-1}} \|\theta\|_K^{-p}\|\theta\|_L^{-n+p}\, d\theta.$$

We also use an elementary formula for the volume of a body K, which can be derived by writing the volume integral in polar coordinates:

$$\mathrm{vol}_n(K) = \frac{1}{n}\int_{S^{n-1}} \|\theta\|_K^{-n}\, d\theta. \tag{3}$$

We are now ready to prove our main result.

Proof of Theorem 1. Since every function from $C^{(n-4)}(S^{n-1})$ can be approximated by an infinitely differentiable function on the sphere (in the $C^{(n-4)}$-norm), it is sufficient to prove the theorem for infinitely smooth bodies K and L.

By Lemma 2, formula (1) and the connection between differentiation and the Fourier transform,

$$(2\pi)^n \int_{S^{n-1}} \|x\|_K^{-1} \|x\|_L^{-n+1}\ dx =$$

$$(2\pi)^n \int_{S^{n-1}} \left(|x|_2^{-n+4}\|x\|_K^{-1}\right)\left(|x|_2^{n-4}\|x\|_L^{-n+1}\right)\ dx =$$

$$\int_{S^{n-1}} \left(|x|_2^{-n+4}\|x\|_K^{-1}\right)^\wedge(\theta)\left(|x|_2^{n-4}\|x\|_L^{-n+1}\right)^\wedge(\theta)\ d\theta =$$

$$\int_{S^{n-1}} \left(|x|_2^{-n+4}\|x\|_K^{-1}\right)^\wedge(\theta)\ (-1)^{(n-4)/2}\Delta^{(n-4)/2}S_L(\theta)\ d\theta.$$

By Lemma 1 with $k=n-4$ and $p=3$, $\left(|x|_2^{-n+4}\|x\|_K^{-1}\right)^\wedge$ is a non-negative function on S^{n-1}, so using the condition (2) of the theorem and repeating the above calculation in the reverse order, we get that

$$\int_{S^{n-1}} \|x\|_K^{-1}\|x\|_K^{-n+1}\ dx \le \int_{S^{n-1}} \|x\|_K^{-1}\|x\|_L^{-n+1}\ dx.$$

By (3) and Hölder's inequality,

$$n\mathrm{vol}_n(K) \le \left(\int_{S^{n-1}} \|\theta\|_K^{-n}\ d\theta\right)^{1/n}\left(\int_{S^{n-1}} \|\theta\|_L^{-n}\ d\theta\right)^{(n-1)/n} =$$

$$n(\mathrm{vol}_n(K))^{1/n}(\mathrm{vol}_n(L))^{(n-1)/n},$$

which implies the inequality for the volumes of K and L. $\square$

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