PracticeMe1. Do These Problems Before HW1
If you discover any misprints, please tell your Instructor
PROBLEMS

1. Find $|\mathbf{a}|$, $\mathbf{a}+\mathbf{b}$, and $2\mathbf{a}-3\mathbf{b}$, where
$\mathbf{a}=\mathbf{i}+3\mathbf{j}-\mathbf{k}, \quad \mathbf{b}=\mathbf{i}+7\mathbf{k}.$

2. Find the *dot product* $\mathbf{a}\cdot\mathbf{b}$, where
$\mathbf{a}=-\mathbf{i}-5\mathbf{k}, \quad \mathbf{b}=2\mathbf{i}+10\mathbf{j}-\mathbf{k}.$

3. Determine whether the following vectors are orthogonal, parallel, or neither.
$\mathbf{a}=\mathbf{i}-2\mathbf{j}+\mathbf{k}, \quad \mathbf{b}=-3\mathbf{i}+6\mathbf{j}-3\mathbf{k}.$

4. Find the values of x and y such that the following vectors are orthogonal:
$\mathbf{a}=(x-1)\mathbf{i}+2y\mathbf{j}, \quad \mathbf{b}=2(x-1)\mathbf{i}+3y\mathbf{j}.$

5. Find the *vector projection* and *scalar projection* of $\mathbf{b}$ onto $\mathbf{a}$, where
$\mathbf{a}=\mathbf{i}+\mathbf{j}-3\mathbf{k}, \quad \mathbf{b}=2\mathbf{i}-\mathbf{k}.$

6. Find the *cross product* $\mathbf{a}\times\mathbf{b}$, where
$\mathbf{a}=2\mathbf{i}+\mathbf{j}-\mathbf{k}, \quad \mathbf{b}=\mathbf{i}-2\mathbf{j}.$

7. Find a vector orthogonal to the plane that contains the vectors $\mathbf{a}=\langle 1,0,1\rangle$ and $\mathbf{b}=\langle 0,-1,3\rangle$ whose tails are placed at the origin.

8. Show that $\mathbf{a}\times\mathbf{3a}=\mathbf{0}$ for all vectors $\mathbf{a}$.

9. Find the area of the parallelogram spanned by $\mathbf{a}=\langle 1,2,0\rangle$ and $\mathbf{b}=\langle 0,1,-3\rangle$. (We say that a parallelogram is spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$ if their tails are placed at the origin, and they serve as two sides of the parallelogram.)

10. Find the area of the parallelogram with vertices at $A(0,0,3)$, $B(1,0,3)$, $C(2,2,3)$, and $D(1,2,3)$.

11. Show that the area of the parallelogram spanned by the vectors $\mathbf{a}=\langle a_1,a_2\rangle$ and $\mathbf{b}=\langle b_1,b_2\rangle$ in the plane is equal to the absolute value of $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = |a_1b_2-a_2b_1|$.

12. What is the volume of the parallelepiped spanned by $\mathbf{a}=\mathbf{i}+\mathbf{j}-\mathbf{k}$, $\mathbf{b}=2\mathbf{i}+\mathbf{j}-2\mathbf{k}$, and $\mathbf{c}=\mathbf{j}-3\mathbf{k}$? (A parallelepiped one of whose vertices is at the origin is said to be spanned by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ if their tails are placed at the origin, and they serve as the sides of the parallelepiped.)

13. Find vector and parametric equations of the line passing through the point $P = (2, 5 - 4)$ and parallel to the vector $\langle 2, -1, 3 \rangle$.

14. Find parametric equations of the line passing through $P = (2, 5-4)$ and $Q = (1, 2, 3)$.

15. Show that the line through $P_1 = (1, 2, 5)$ and $Q_1 = (0, 0, 1)$ is parallel to the line through $P_2 = (1, 1, 2)$ and $Q_2 = (3, 5, 10)$.

16. Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting, where

$$L_1 \;:\; \frac{x-2}{3} = \frac{y-1}{2} = \frac{z+3}{-2}, \qquad L_2 \;:\; \frac{x+1}{2} = y - 1 = \frac{z-3}{4}.$$

17. Find an equation of the plane passing through $P = (2, 3, 0)$ and with a normal vector $\mathbf{n} = \langle 2, -1, 1 \rangle$.

18. Find an equation of the plane passing through $P = (1, 2-3)$ and parallel to the plane $x - 2y + 3z - 10 = 0$.

19. Find an equation of the plane through $P = (1, 2-3)$, $Q = (1, 1, 5)$, and $R = (2, 0, -3)$.

20. Find an equation of the plane that passes through $P = (1, 1, 5)$, and contains the line $x = 1 + 2t,\ y = 3 - 4t,\ z = 2t$.

21. Find the point at which the line $L[t] = (3 + 2t)\mathbf{i} + (-1 + 2t)\mathbf{j} + (5 - t)\mathbf{k}$ intersects the plane $2x - 3y + 4z - 10 = 0$.

22. Determine whether the planes $2x + y + 2z - 1 = 0$ and $2x - 4y + z - 2 = 0$ are parallel, perpendicular, or neither.

23. Find the angle between the planes $x + y + 2z - 9 = 0$ and $2x - 4y + 6z - 20 = 0$.

24. Find an equation of the plane consisting of all points equidistant from $P = (1, 2, 3)$ and $Q = (1, 4, 1)$.

25. Find an equation of the plane that passes through $P = (1, 2, 3)$ and through the line of intersection of the planes $x - y + z - 1 = 0$ and $-x + y + z - 3 = 0$.

SOLUTIONS

Solution 1:

$|\mathbf{a}| = \sqrt{(1)^2 + (3)^2 + (-1)^2} = \sqrt{11}$; $\quad \mathbf{a} + \mathbf{b} = (1+1)\mathbf{i} + (3+0)\mathbf{j} + (-1+7)\mathbf{k} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$; $2\mathbf{a} - 3\mathbf{b} = (-1)\mathbf{i} + 6\mathbf{j} - 23\mathbf{k}$.

Solution 2:

$\mathbf{a} \cdot \mathbf{b} = (-1)(2) + (0)(10) + (-5)(-1) = 3.$

Solution 3:
$\mathbf{a}\cdot\mathbf{b} = (1)(-3) + (-2)(6) + (1)(-3) = -18 \neq 0$, so that $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. Note that $\mathbf{a} = -3\mathbf{b}$ which means that $\mathbf{a}$ is parallel to $\mathbf{b}$.

Solution 4:
$\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a}\cdot\mathbf{b} = 0$. We have $\mathbf{a}\cdot\mathbf{b} = 2(x-1)^2 + 6y^2 = 0$. From this we see that $x - 1 = 0$ and $y = 0$.
Answer: $x = 1$ and $y = 0$.

Solution 5: The *vector projection* of $\mathbf{b}$ onto $\mathbf{a}$ (the same as the *push of* $\mathbf{b}$ *along* $\mathbf{a}$), is defined by

$$\mathbf{Proj_a}(\mathbf{b}) = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a}.$$

We have

$$\mathbf{Proj_a}(\mathbf{b}) = \frac{(1)(2) + (1)(0) + (-3)(-1)}{(1)^2 + (1)^2 + (-3)^2}(\mathbf{i}+\mathbf{j}-3\mathbf{k}) = \frac{5}{11}(\mathbf{i}+\mathbf{j}-3\mathbf{k}).$$

The *scalar projection* of $\mathbf{b}$ onto $\mathbf{a}$ is defined by

$$\mathrm{Proj}_{\mathbf{a}}(\mathbf{b}) = ||\mathbf{b}||\cos(\theta) = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||} = \frac{5}{\sqrt{11}}.$$

Solution 6:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -2 & 0 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= \mathbf{i}((1)(0) - (-1)(-2)) - \mathbf{j}((2)(0) - (-1)(1)) + \mathbf{k}((2)(-2) - (1)(1))$$

$$= -2\mathbf{i} - \mathbf{j} - 5\mathbf{k}.$$

Solution 7: The cross product $\mathbf{a}\times\mathbf{b}$ is orthogonal to the plane that contains $\mathbf{a}$ and $\mathbf{b}$. We have

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & -1 & 3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 0 & 1 \\ -1 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

$$= \mathbf{i}((0)(3) - (-1)(1)) - \mathbf{j}((1)(3) - (1)(0)) + \mathbf{k}((1)(-1) - (0)(0))$$

$$= \mathbf{i} - 3\mathbf{j} - \mathbf{k}.$$

Solution 8: Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, then $3\mathbf{a} = \langle 3a_1, 3a_2, 3a_3\rangle$. We have

$$\mathbf{a} \times \mathbf{3a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ 3a_1 & 3a_2 & 3a_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ 3a_2 & 3a_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ 3a_1 & 3a_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ 3a_1 & 3a_2 \end{vmatrix}$$

$$= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \langle 0, 0, 0\rangle.$$

Solution 9: The area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ equals $||\mathbf{a} \times \mathbf{b}||$. We have

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 0 & 1 & -3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & 0 \\ 1 & -3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= -6\mathbf{i} + 3\mathbf{j} + \mathbf{k}.$$

Then the area of the parallelogram is $||-6\mathbf{i} + 3\mathbf{j} + \mathbf{k}|| = \sqrt{(-6)^2 + (3)^2 + (1)^2} = \sqrt{46}$.

Solution 10: The parallelogram has $\overrightarrow{AB}$ and $\overrightarrow{AD}$ (placed so that their tales are at A) as its sides . The area of the parallelogram is $||\overrightarrow{AB} \times \overrightarrow{AD}||$. We have $\overrightarrow{AB} = \langle (1-0), (0-0), (3-3)\rangle = \langle 1, 0, 0\rangle$ and $\overrightarrow{AD} = \langle (1-0), (2-0), (3-3)\rangle = \langle 1, 2, 0\rangle$. Then

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{vmatrix} = \mathbf{k}\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = \langle 0, 0, 2\rangle.$$

The area of the parallelogram is $||\langle 0, 0, 2\rangle|| = 2$.

Solution 11: We set $\mathbf{a} = \langle a_1, a_2, 0\rangle$ and $\mathbf{b} = \langle b_1, b_2, 0\rangle$ and consider them as vectors in space. Then the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ is $||\mathbf{a} \times \mathbf{b}||$, where

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Thus the area of the parallelogram is $||\langle 0, 0, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\rangle|| = \sqrt{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2} = |a_1b_2 - a_2b_1|$.

Solution 12: The volume of the parallelepiped spanned by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is equal to $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$. (The scalar $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is called the *scalar triple product* of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.) Let us find $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. First we get

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 1 & -2 \\ 1 & -3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -2 \\ 0 & -3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= -\mathbf{i} + 6\mathbf{j} + 2\mathbf{k} = \langle -1, 6, 2\rangle.$$

Then $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \langle 1, 1, -1\rangle \cdot \langle -1, 6, 2\rangle = (1)(-1) + (1)(6) + (-1)(2) = 3.$
The volume of the parallelepiped is $|3| = 3$.
Remark. You can use another formula for the volume of the parallelepiped spanned by $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3\rangle$: The volume is equal to the absolute value of $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Solution 13: The vector equation of the line through $P = (p_1, p_2, p_3)$ parallel to $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ is given by

$$\mathrm{L}[t] = P + t\ \mathbf{a} = (p_1 + ta_1)\mathbf{i} + (p_2 + ta_2)\mathbf{j} + (p_3 + ta_3)\mathbf{k},\ -\infty < t < \infty.$$

The parametric equations are

$$x(t) = p_1 + ta_1, \quad y(t) = p_2 + ta_2, \quad z(t) = p_3 + ta_3.$$

We have: $\mathrm{L}[t] = (2+2t)\mathbf{i}+(5-t)\mathbf{j}+(-4+3t)\mathbf{k}$. In the parametric form: $x(t) = 2+2t,\ y(t) = 5 - t,\ z(t) = -4 + 3t$.

Solution 14: The line is passing through $(2, 5, -4)$ and parallel to the vector

$$\overrightarrow{PQ} = \langle 1 - 2,\ 2 - 5,\ 3 - (-4)\rangle = \langle -1, -3, 7\rangle.$$

The vector equation is given by (see Problem 13)

$$\mathrm{L}[t] = (2 - t)\mathbf{i} + (5 - 3t)\mathbf{j} + (-4 + 7t)\mathbf{k}.$$

From this we get the parametric equations

$$x(t) = 2 - t, \quad y(t) = 5 - 3t, \quad z(t) = -4 + 7t,\ -\infty < t < \infty.$$

Solution 15: Denote the first line by L_1 and the second one by L_2. Note that L_1 is parallel to $\overrightarrow{P_1Q_1} = \langle -1, -2, -4\rangle$, and L_2 is parallel to $\overrightarrow{P_2Q_2} = \langle 2, 4, 8\rangle$. It is easy to see that $\overrightarrow{P_2Q_2} = -2\overrightarrow{P_1Q_1}$. Hence $\overrightarrow{P_2Q_2}$ is parallel to $\overrightarrow{P_1Q_1}$, and so the lines are parallel.

Remark. You can also check that $\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2} = \mathbf{0}$ which means that $\overrightarrow{P_1Q_1}$ is parallel to $\overrightarrow{P_2Q_2}$.

Answer: L_1 and L_2 are parallel.

Solution 16: Let us have the equations of L_1 and L_2 in a parametric form. For L_1 we set $\frac{x-2}{3} = \frac{y-1}{2} = \frac{z+3}{-2} = t$. Solving these equations for x, y, z we get:

$$L_1 \ : \ x = 2 + 3t, \quad y = 1 + 2t, \quad z = -3 - 2t, \quad -\infty < t < \infty.$$

For L_2 we set $\frac{x+1}{2} = y - 1 = \frac{z-3}{4} = s$. Solving these equations for x, y, z we get:

$$L_2 \ : \ x = -1 + 2s, \quad y = 1 + s, \quad z = 3 + 4s, \quad -\infty < s < \infty.$$

We see that L_1 is parallel to $\mathbf{a} = \langle 3, 2, -2\rangle$, and L_2 is parallel to $\mathbf{b} = \langle 2, 1, 4\rangle$. Note that $\mathbf{a}$ is not parallel to $\mathbf{b}$. (Check that $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$!) Hence L_1 and L_2 are not parallel. Also L_1 and L_2 are not intersecting. (Equating the corresponding x's, y's, and z's we get: $2 + 3t = -1 + 2s, \ 1 + 2t = 1 + s, \ -3 - 2t = 3 + 4s$. These three simultaneous equations have no common solution in t and s.)

Answer: L_1 and L_2 are skew.

Solution 17: If a plane passes through $P = (p_1, p_2, p_3)$, and its normal vector $\mathbf{n} = \langle n_1, n_2, n_3\rangle$ is given, then the plane has the following equation:

$$n_1(x - p_1) + n_2(y - p_2) + n_3(z - p_3) = 0.$$

Thus the equation is $2(x - 2) + (-1)(y - 3) + (1)(z - 0) = 0$, or after simplification $2x - y + z - 1 = 0$.

Solution 18: From the equation $x - 2y + 3z - 10 = 0$ we get a normal vector $\mathbf{n} = \langle 1, -2, 3\rangle$ for the given plane. (Just write out the coefficients at x, y, and z.) Since the planes are parallel, it is also normal to the unknown plane. Then by the preceding solution we get an equation for the plane through $(1, 2, -3)$ with a normal vector $\mathbf{n}$:

$$(1)(x-1) + (-2)(y-2) + (3)(z-(-3)) = 0, \qquad \text{or after simplification} \quad x - 2y + 3z + 12 = 0.$$

Solution 19: A normal vector to the plane should be perpendicular to two vectors in the plane, say $\overrightarrow{PQ} = \langle 0, -1, 8\rangle$ and $\overrightarrow{PR} = \langle 1, -2, 0\rangle$. We can choose $\mathbf{n} = \langle 0, -1, 8\rangle \times \langle 1, -2, 0\rangle$ as a normal vector. We have

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 8 \\ 1 & -2 & 0 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 8 \\ -2 & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 0 & 8 \\ 1 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 0 & -1 \\ 1 & -2 \end{vmatrix}$$

$$= 16\mathbf{i} + 8\mathbf{j} + \mathbf{k}.$$

Then the equation of the plane through $(1, 2 - 3)$ with a normal vector $\mathbf{n} = \langle 16, 8, 1\rangle$ is

$$16(x - 1) + 8(y - 2) + (z + 3) = 0.$$

Solution 20: Since the line lies on the plane, the point $Q = (1, 3, 0)$ and the direction vector $\mathbf{a} = \langle 2, -4, 2\rangle$ lie on the plane. Also the plane passes through P and Q, so the vector $\overrightarrow{PQ} = \langle 0, 2, -5\rangle$, is on the plane. Then we can choose $\mathbf{a} \times \overrightarrow{PQ}$ as a normal vector $\mathbf{n}$. We have

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 2 \\ 0 & 2 & -5 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -4 & 2 \\ 2 & -5 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 2 \\ 0 & -5 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix}$$

$$= 16\mathbf{i} + 10\mathbf{j} + 4\mathbf{k}.$$

Then the equation of the plane through $(1, 1, 5)$ with a normal vector $\mathbf{n} = \langle 16, 10, 4\rangle$ is given by $16(x-1) + 10(y-1) + 4(z-5) = 0$.

Solution 21: Substitute the parametric equations of the line $x = 3+2t,\ y = -1+2t,\ z = 5-t$ into the equation of the plane and solve for t:

$$2(3+2t) - 3(-1+2t) + 4(5-t) - 10 = 0, \quad 19 - 6t = 0, \quad t = 19/6.$$

Then $x = 3 + 2(19/6), \quad y = -1 + 2(19/6), \quad z = 5 - (19/6).$

Answer: $(28/3,\ 16/3,\ 11/6)$.

Solution 22: We first find the normal vectors for the given planes, $\mathbf{n_1} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, and $\mathbf{n_2} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$. Then $\mathbf{n_1} \cdot \mathbf{n_2} = (2)(2) + (1)(-4) + (2)(1) = 2 \neq 0$. Hence the planes are not perpendicular. Also they are not parallel because

$$\mathbf{n_1} \times \mathbf{n_2} = (2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \times (2\mathbf{i} - 4\mathbf{j} + \mathbf{k}) = 7\mathbf{i} + 2\mathbf{j} - 10\mathbf{k} \neq \mathbf{0}.$$

Answer: The planes are neither parallel, nor perpendicular.

Solution 23: We find the normal vectors for the given planes, $\mathbf{n_1} = \langle 1, 1, 2\rangle$, and $\mathbf{n_2} = \langle 2, -4, 6\rangle$. If θ is the angle between $\mathbf{n_1}$ and $\mathbf{n_2}$, then

$$\cos(\theta) = \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{||\mathbf{n_1}||\ ||\mathbf{n_2}||} = \frac{(1)(2) + (1)(-4) + (2)(6)}{\sqrt{(1)^2 + (1)^2 + (2)^2}\sqrt{(2)^2 + (-4)^2 + (6)^2}} = \frac{10}{\sqrt{6}\sqrt{56}} = \frac{5}{2\sqrt{21}}.$$

Answer: The angle between the planes is $\arccos\left(5/(2\sqrt{21})\right)$

Solution 24: The plane passes through the midpoint $R = ((1+1)/2,\ (2+4)/2,\ (3+1)/2) = (1, 3, 2)$ of PQ and is perpendicular to $\overrightarrow{PQ} = \langle 1-1, 4-2, 1-3\rangle = \langle 0, 2, -2\rangle$. We can choose $\overrightarrow{PQ}$ as a normal vector and get the equation $(0)(x-1) + (2)(y-3) + (-2)(z-2) = 0$.

Answer: $y - z - 1 = 0$.

Solution 25: Let us find two points on the line of intersection of $x - y + z - 1 = 0$ and $-x + y + z - 3 = 0$. We can set $x = 0$; then solving the equations $-y + z - 1 = 0$ and $y + z - 3 = 0$ we get $y = 1,\ z = 2$. Thus $Q = (0, 1, 2)$ lies on the line of intersection. Similarly setting $y = 0$ we get the point $R = (-1, 0, 2)$ on the line of intersection. Now we can get an equation of the plane passing through P, Q, and R as in Problem 19. We choose $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ as a normal vector. Then $\mathbf{n} = -\mathbf{i} + \mathbf{j}$. Since the plane passes through $(1, 2, 3)$ we get $(-1)(x - 1) + (1)(y - 2) + (0)(z - 3) = 0$.

Answer: $-x + y - 1 = 0$.