PracticeMe3. Do these Problems Before HW3
If you discover any misprints, please tell your Instructor

PROBLEMS

Problem 1. Find the directional derivative of the function $f(x,y) = x^2y^3 - 4y$ at the point (2,-1) in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

Problem 2. If $f(x,y,z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at (1,3,0) in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Problem 3. (a) If $f(x,y) = xe^y$, find the rate of change of f at the point $P(2,0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

Problem 4. Find the equation of the tangent plane at the point (-2,1,-3) to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

Problem 5. Find the local extrema of $f(x,y) = x^4 + y^4 - 4xy + 1$.

Problem 6. Find the shortest distance from the point (1,0,-2) to the plane $x + 2y + z = 4$.

Problem 7. Find the absolute maximum and minimum values of the function $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x,y)|0 \leq x \leq 3, 0 \leq y \leq 2\}$.

Problem 8. For the function $f(x,y) = \ln(x/y)$ find the maximum value of the directional derivative at the point (1,2) and give the direction that produces this maximum value.

Problem 9. In what direction from the point (1,1) does the temperature $T(x,y) = \frac{100}{1+x^2+y^2}$ change most rapidly, and what is this maximum rate of change?

Problem 10. (a) Find the direction in which the function

$$f(x,y) = x^2 - y^2 + x + y + 5$$

increases the most at the point {1,5}.

(b) What is this maximum rate of increase at this point?

Problem 11. Find the absolute extrema and where they occur for the function

$$f(x,y) = x^2 + y^2 - 10x - 6y + 41$$

Problem 12. Find all critical points of

$$f(x,y) = x^2 + y^2 + xy + x + 2$$

Problem 13. Find all local maximum and minimum values of the function f defined by

$$f(x,y) = x^2 - 2xy + 4y^2 - 2x - 4y + 1$$

Problem 14. Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$

Problem 15. Maximize $f(x,y) = 6 - 2x^2 - y^2$ subject to the constraint $xy = 2$

Problem 16. Find the minimum and the maximum values of the function $f(x,y) = x^2 - y^2$ subject to the constraint $g(x,y) = x^2 + y^2 = 1$

Problem 17. Classify all critical points of f

$$f(x,y) = 5 + 4x - 2x^2 + 3y - y^2$$

Problem 18. Go with $f(x,y,z) = x^2 - 3y^2 + xz$. In which direction you should leave the point {1,2,3} to get the greatest possible increase of $f(x,y,z)$?

Problem 19. Go with

$$f(x,y,z) = x^2 + y^2 - 2xyz$$

In which direction you should leave the point (1,0,1) to get the greatest possible decrease of $f(x,y,z)$?

Problem 20. Find the equation of the tangent plane to the surface $f(x,y) = 16 - x^2 - y^2$ at the point (1,2).

Problem 21. Suppose that the temperature $T(x,y,z)$ at point (x,y,z) is given by

$$T(x,y,z) = 2x^2 + 3y^2 - 4z.$$

Find the rate of change of T at the point $P = (1,1,1)$ in the direction from P to $Q = \{2,2,5\}$.

Problem 22. Classify all critical points of

$$f(x,y) = x^2 + 3y - y^3$$

SOLUTIONS

Solution 1.

$$\begin{aligned} \text{grad} f(x,y) &= 2xy^3\mathbf{i} + (3x^2y^2 - 4)\mathbf{j} = \{2xy^3, 3x^2y^2 - 4\} \\ \text{grad} f(2,-1) &= -4\mathbf{i} + 8\mathbf{j} = \{-4, 8\} \end{aligned}$$

Note that $\mathbf{v}$ is not a unit vector, so a unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}$$

Therefore,

$$D_u f(2,-1) = \text{grad} f(2,-1) \cdot \mathbf{u} = \{-4, 8\} \cdot \left\{\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right\} = \frac{32}{\sqrt{29}}$$

Solution 2. (a) The gradient of f is

$$\text{grad} f(x,y,z) = \nabla f(x,y,z) = \{\sin yz, xz\cos yz, xy\cos yz\}$$

(b) At (1,3,0) we have $\nabla f(1,3,0) = \{0,0,3\}$. The unit vector in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is

$$\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k} = \left\{\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right\}$$

Therefore,

$$\begin{aligned} D_u f(1,3,0) &= \nabla f(1,3,0)\cdot \mathbf{u} \\ &= \{0,0,3\}\cdot\left\{\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right\} \\ &= 3\left(-\frac{1}{\sqrt{6}}\right) = -\sqrt{\frac{3}{2}} \end{aligned}$$

Solution 3. (a)

$$\begin{aligned} \text{grad} f(x,y) = \nabla f(x,y) &= \{e^y, xe^y\} \\ \nabla f(2,0) &= \{1,2\} \end{aligned}$$

The unit vector in the direction of $\overrightarrow{PQ} = \{-1.5, 2\}$ is $\mathbf{u} = \{-\frac{3}{5}, \frac{4}{5}\}$ so the rate of change of f in the direction from P to Q is

$$\begin{aligned} D_u f(2,0) &= \nabla f(2,0)\cdot \mathbf{u} = \{1,2\}\cdot\left\{-\frac{3}{5}, \frac{4}{5}\right\} \\ &= 1\left(-\frac{3}{5}\right) + 2\left(\frac{4}{5}\right) = 1 \end{aligned}$$

(b) The function f increases fastest in the direction of the gradient vector $u = \frac{\nabla f(2,0)}{|\nabla f(2,0)|} = \{\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\}$. The maximum rate of change is

$$|\nabla f(2,0)| = |\{1,2\}| = \sqrt{5}$$

Solution 4. Let

$$\begin{aligned} F(x,y,z) &= \frac{x^2}{4} + y^2 + \frac{z^2}{9} \\ \text{grad} F(x,y,z) &= \left\{\frac{x}{2}, 2y, \frac{2z}{9}\right\} \\ \text{grad} F(-2,1,-3) &= \{-1, 2, -\frac{2}{3}\} \end{aligned}$$

The equation of the plane is

$$-1(x+2) + 2(y-1) - \frac{2}{3}(z+3) = 0$$

or $3x - 6y + 2z + 18 = 0$.

Solution 5.

$$\frac{\partial f}{\partial x} = 4x^3 - 4y; \quad \frac{\partial f}{\partial y} = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain the equations

$$x^3 - y = 0 \quad \text{and} \quad y^3 - x = 0$$

Solving with Mathematica we find three real roots: $x = 0, 1, -1$. The three critical points are (0,0), (1,1), and (-1,-1).

Next we calculate the second partial derivatives:

$$f_{xx} = 12x^2 \quad f_{xy} = -4 \quad f_{yy} = 12y^2$$
$$D(x,y) = \text{hessiandet}(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

Since $D(0,0) = -16 < 0$, it follows from the Second Derivatives test that the origin is a saddle point. Since $D(1,1) = 128 > 0$ and $f_{xx}(1,1) = 12 > 0$, we see that $f(1,1) = -1$ is a local minimum. Similarly, we have $D(-1,-1) = 128 > 0$ and $f_{xx}(-1,-1) = 12 > 0$, so $f(-1,-1) = -1$ is also a local minimum.

Solution 6. The distance from any point (x, y, z) to the point (1,0,-2) is

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$$

but if (x, y, z) lies on the plane $x + 2y + z = 4$, then $z = 4 - x - 2y$ and so we have $d = \sqrt{(x-1)^2 + y^2 + (6 - x - 2y)^2}$. We can minimize d by minimizing the simpler expression

$$d^2 = f(x,y) = (x-1)^2 + y^2 + (6 - x - 2y)^2$$

By solving the equations

$$f_x = 2(x-1) - 2(6 - x - 2y) = 4x + 4y - 14 = 0$$
$$f_y = 2y - 4(6 - x - 2y) = 4x + 10y - 24 = 0$$

we find that the only critical point is $(\frac{11}{6}, \frac{5}{3})$. Since $f_{xx} = 4$, $f_{xy} = 4$, and $f_{yy} = 10$, we have at this point $f_{xx}f_{yy} - (f_{xy})^2 = 24 > 0$. Since $f_{xx} > 0$, by the Second Derivatives Test f has a local minimum at $(\frac{11}{6}, \frac{5}{3})$.

$$d = \sqrt{(x-1)^2 + y^2 + (6 - x - 2y)^2} = \sqrt{\frac{25}{6}}$$

The shortest distance from (1,0,-2) to the plane $x + 2y + z = 4$ is $\sqrt{\frac{25}{6}}$.

Solution 7. Since f is a polynomial, it is continuous on the closed, bounded rectangle D, so there is both an absolute maximum and an absolute minimum. Solving

$$f_x = 2x - 2y = 0 \quad \text{and} \quad f_y = -2x + 2 = 0$$

gets that the only critical point is (1,1), and the value of f there is $f(1,1) = 1$.

Let's look at the values of f on the boundary of D. The boundary of D consists of the four line segments L_1, L_2, L_3, and L_4, where L_1 is the line joining {0,0} and {3,0}, L_2 is the line joining {3,0} and {3,2}, L_3 is the line joining {3,2} and {0,2}, and L_4 is the line joining {0,2} and {0,0}.

On L_1 we have $y = 0$ and $f(x,0) = x^2 \quad 0 \le x \le 3$. This is an increasing function of x, so its minimum value is $f(0,0) = 0$ and its maximum value is $f(3,0) = 9$.

On L_2 we have $x = 3$ and $f(3,y) = 9 - 4y \quad 0 \le y \le 2$. This is a decreasing function of y, so its maximum value is $f(3,0) = 9$ and its minimum value is $f(3,2) = 1$.

On L_3 we have $y = 2$ and $f(x,2) = x^2 - 4x + 4 \quad 0 \le x \le 3$. By observing that $f(x,2) = (x-2)^2$, we see that the minimum value of this function is $f(2,2) = 0$ and the maximum value is $f(0,2) = 4$.

Finally, on L_4 we have $x = 0$ and $f(0,y) = 2y \quad 0 \le y \le 2$ with maximum value $f(0,2) = 4$ and minimum value $f(0,0) = 0$. Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

Compare these values with the value $f(1,1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3,0) = 9$ and the absolute minimum value is $f(0,0) = f(2,2) = 0$.

Graph this function with Mathematica to confirm your solution.

Solution 8.

$$\text{grad} f(x,y) = \left\{\frac{1}{x}, -\frac{1}{y}\right\} \quad \text{so} \quad \text{grad} f(1,2) = \left\{1, -\frac{1}{2}\right\}$$

So the maximum value of the directional derivative is

$$\|\text{grad} f(1,2)\| = \left|\left\{1, -\frac{1}{2}\right\}\right| = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$$

This maximum value occurs in the direction of the gradient vector, which is $u = \left\{\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right\}$

Solution 9.

$$\text{grad} T(x,y) = \left\{-\frac{200x}{(1+x^2+y^2)^2}, -\frac{200y}{(1+x^2+y^2)^2}\right\}$$

The temperature $T(x,y)$ changes most rapidly in the direction of $\frac{\text{grad} T(1,1)}{\|\text{grad} T(1,1)\|} = \left\{-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$ and the maximum rate of change is

$$\|\text{grad} T(1,1)\| = \frac{200\sqrt{2}}{9}$$

Solution 10. (a) In the direction of $\text{grad} f(1,5)$

$$\text{grad} f(x,y) = \{1 + 2x, 1 - 2y\}$$
$$\text{grad} f(1,5) = \{3, -9\}.$$

(b) Maximum rate of increase at this point $= \|\text{grad} f(1,5)\| = 3\sqrt{10}$.

Solution 11. Notice that

$$f(x,y) = (x-5)^2 + (y-3)^2 + 7$$

this shows that $f(x,y) \geq 7$ for all x and y. So f has only an absolute minimum at {5,3}, it does not have an absolute maximum since $f(x,y)$ becomes very huge for either $|x|$ or $|y|$ near infinity.

Solution 12. The only critical points are the ones that make $\text{grad} f = 0$. So $\text{grad} f(x,y) = \{1+2x+y, x+2y\}$. Solving $\text{grad} f(x,y) = 0$ yields $x = -\frac{2}{3}$ and $y = \frac{1}{3}$. So $\{-\frac{2}{3}, \frac{1}{3}\}$ is the only critical point.

Solution 13. First, we find the critical points by setting $\text{grad} f(x,y) = 0$.

$$\text{grad} f(x,y) = \{2x - 2y - 2, -2x + 8y - 4\}$$

$\Rightarrow x = 2, y = 1$. To test this point, {2,1}, we need the second partials:

$$f_{xx} = 2 \quad f_{xy} = -2 \quad f_{yy} = 8$$

So $D(x,y) = \text{hessiandet}(x,y) = 2(8) - (-2)^2 = 16 - 4 = 12$. Since $D(x,y) > 0$ for all x and y, we see that $D(2,1) > 0$. Since $f_{xx}(2,1) = 2 > 0$, we conclude that f has a local minimum at (2,1). This minimum value is $f(2,1) = -3$.

Solution 14. Use Lagrange's multipliers to solve two equations

$$\text{eqn1} = g(x,y) = x^2 + y^2 = 1$$
$$\text{eqn2} = \text{grad} f(x,y) = \lambda \text{grad} g(x,y).$$

Actually eqn2 gives two equations since

$$\{2x, 4y\} = \{2\lambda x, 2\lambda y\}$$

is equivalent to

$$2x = 2\lambda x$$
$$4y = 2\lambda y$$

So we have now three equations to solve for three unknown λ, x and y.

$$2x = 2x\lambda$$
$$4y = 2\lambda y$$
$$x^2 + y^2 = 1$$

The first equation gives $x = 0$ or $\lambda = 1$. If $x = 0$, the third equation gives that $y = \pm 1$. If $\lambda = 1$, then $y = 0$ from the second equation and the third equation gives $x = \pm 1$. Therefore the possible extreme values of f occurs at the points (0,1), (0,-1), (1,0) and (-1,0). $f(0,1) = 2$, $f(0,-1) = 2$, $f(1,0) = 1$ and $f(-1,0) = 1$.

Hence the maximum value of f on the circle $x^2 + y^2 = 1$ is 2 which occur at the points (0,1) and (0,-1) and the minimum value of f is 1 which occurs at (1,0) and (-1,0).

Solution 15.

$$\begin{aligned}\text{eqn1} &= g(x,y) = xy = 2\\ \text{eqn2} &= \text{grad} f(x,y) = \lambda \text{grad} g(x,y)\end{aligned}$$

The above will generate three equations

$$\begin{aligned} xy &= 2\\ \{-4x, -2y\} &= \lambda\{y, x\}\end{aligned}$$

Therefore

$$\begin{aligned} xy &= 2\\ -4x &= \lambda y\\ -2y &= \lambda x\end{aligned}$$

Solving these equations by hand or using Mathematica gives

$$\begin{aligned} &\lambda = -2\sqrt{2} && x = -2^{1/4} && y = -2^{3/4}\\ &\lambda = -2\sqrt{2} && x = 2^{1/4} && y = 2^{3/4}\end{aligned}$$

Evaluating f at these points gives

$$\begin{aligned} f(-2^{1/4}, -2^{3/4}) &= 6 - 4\sqrt{2}\\ f(2^{1/4}, 2^{3/4}) &= 6 - 4\sqrt{2}\end{aligned}$$

This is a maximum since if one takes the point (2,1) which satisfies $xy = 2$, one finds that $f(2,1) = -3$.

Solution 16.

$$\begin{aligned} x^2 + y^2 &= 1\\ \{2x, -2y\} &= \lambda\{2x, 2y\}\end{aligned}$$

Hence we have to solve

$$\begin{aligned} x^2 + y^2 &= 1\\ 2x &= 2\lambda x\\ -2y &= 2\lambda y\end{aligned}$$

These have the following solutions

$$\begin{aligned} &\lambda = -1 && y = -1 && x = 0\\ &\lambda = -1 && y = 1 && x = 0\\ &\lambda = 1 && x = -1 && y = 0\\ &\lambda = 1 && x = 1 && y = 0\end{aligned}$$

To get these solutions, you can use Mathematica or you can do them by hand as follows: The second equation gives

$$x = 0 \quad \text{or} \quad \lambda = 1$$

The third equation gives

$$y = 0 \quad \text{or} \quad \lambda = -1$$

$$\begin{aligned} &\text{If} \quad \lambda = 1 \Rightarrow y = 0 \Rightarrow x = \pm 1 \\ &\text{If} \quad \lambda = -1 \Rightarrow x = 0 \Rightarrow y = \pm 1. \end{aligned}$$

Evaluating f at the points above gives

$$\begin{aligned} &f(0, -1) = -1 = f(0, 1) \\ &f(-1, 0) = 1 = f(1, 0) \end{aligned}$$

So you conclude that f has a minimum at (0,1) and (0,-1). This minimum is equal -1 and f has a maximum at (-1,0) and (1,0) and this maximum is equal to 1.

Solution 17.

$$\text{grad} f(x, y) = \{4 - 4x, 3 - 2y\}$$

the gradient is zero at $x = 1$, $y = 3/2$.

$$\begin{aligned} \text{hessiandet}(x, y) &= f_{xx} f_{yy} - f_{xy}^2 \\ &= (-4)(-2) = 8 \end{aligned}$$

so hessiandet is constant and is equal to 8 at any point. To find out if f has a maximum or minimum at the point $\{1, 3/2\}$, we compute $f_{xx}(1, 3/2)$. But $f_{xx}(x, y) = -4$ for any point hence f has a maximum at $\{1, 3/2\}$.

Solution 18. You should leave in the direction of the gradient at {1,2,3}.

$$\begin{aligned} &\text{grad} f(x, y, z) = \{2x + z, -6y, x\} \\ &\text{grad} f(1, 2, 3) = \{5, -12, 1\} \end{aligned}$$

So one should leave the point {1,2,3} in the direction of the vector {5,-12,1}.

Solution 19.

$$\begin{aligned} &\text{grad} f(x, y, z) = \{2x - 2yz, 2y - 2xz, -2xy\} \\ &\text{grad} f(1, 0, 1) = \{2, -2, 0\} \end{aligned}$$

One should leave the point (1,0,1) in the direction of

$$-\text{grad} f(1, 0, 1) = \{-2, 2, 0\}$$

Solution 20.

$$z = f(a,b) + \text{grad}f(a,b)\cdot\{x-a, y-b\}$$
$$z = f(1,2) + \text{grad}f(1,2)\cdot\{x-1, y-2\}$$

$$\begin{aligned}\text{grad}f(x,y) &= \{-2x,-2y\}\\ \text{grad}f(1,2) &= \{-2,-4\}\\ f(1,2) &= 16-1-4 = 11\end{aligned}$$

So

$$\begin{aligned}z &= 11 + \{-2,-4\}\cdot\{x-1,y-2\}\\ z &= 11 + ((-2)(x-1) + (-4)(y-2))\\ z &= 21 - 2x - 4y\end{aligned}$$

Solution 21. Let us find a unit vector in the direction of PQ

$$U = \{1,1,4\}/\sqrt{18}$$

Answer= $\text{grad}T(1,1,1)\cdot U$

$$\begin{aligned}\text{grad}T(x,y,z) &= \{4x,6y,-4\}\\ \text{grad}T(1,1,1) &= \{4,6,-4\}\\ \text{grad}T(1,1,1)\cdot U &= -\sqrt{2}\end{aligned}$$

Solution 22.

$$\text{grad}f(x,y) = \{2x, 3-3y^2\}$$
$$\text{grad}f(x,y) = \{0,0\}$$

if $x = 0$ and $y = \pm 1$. We have two critical points (0,1) and (0,-1).

$$\begin{aligned}\text{hessiandet}(x,y) &= f_{xx}f_{yy} - f_{xy}^2\\ &= 2(-6y) - 0 = -12y\end{aligned}$$

$\text{hessiandet}(0,1) = -12 \Rightarrow f$ has a saddle point at (0,1).

Since $\text{hessiandet}(0,-1) = 12$, we have to test the second derivative since $f_{xx}(x,y) = 2$, then $f_{xx}(0,-1) = 2 \Rightarrow f$ has a relative minimum at (0,-1).