PracticeMe4. Do these Problems Before HW4

If you discover some misprints, please report them to your instructor.

Double Integrals, Path Integrals and Green's Theorm

PROBLEMS

1. **Problem:** Evaluate the iterated integral

$$\int_0^3 \int_{-1}^1 \frac{x^2}{1+y^2} dy\ dx$$

2. **Problem:** Evaluate the double integral

$$\iint_R (x+2y) dx dy$$

where R is the rectangular region $\{(x,y) : 0 \le x \le 4, 0 \le y \le 1\}$.

3. **Problem:** Evaluate the following iterated integrals.

(a) $\displaystyle\int_0^2 \int_{x^2}^{4-x} (x-2y) dy\ dx$ (b) $\displaystyle\int_0^1 \int_{y-1}^{\sqrt{1-y^2}} (xy+y) dx\ dy$

(a)

$$\begin{aligned}
\int_0^2 \int_{x^2}^{4-x} (x-2y) dy\ dx &= \int_0^2 \left[xy - y^2\right]_{x^2}^{4-x} dx \\
&= \int_0^2 [x(4-x) - (4-x)^2 - x^3 + x^4] dx \\
&= \int_0^2 [4x - x^2 - (4-x)^2 - x^3 + x^4] dx \\
&= 2x^2 - \frac{x^3}{3} + \frac{(4-x)^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\Bigg]_0^2 = -\frac{164}{15}
\end{aligned}$$

(b)

$$\int_0^1 \int_{y-1}^{\sqrt{1-y^2}} (xy+y)dx\,dy$$

$$\begin{aligned}
&= \int_0^1 \left[\frac{x^2}{2}y + xy\right]_{y-1}^{\sqrt{1-y^2}} dy \\
&= \int_0^1 \left[\frac{(1-y^2)y}{2} + y\sqrt{1-y^2} - \frac{(y-1)^2}{2}y - y(y-1)\right] dy \\
&= \int_0^1 \left[\frac{1}{2}(y-y^3) + y\sqrt{1-y^2} - \frac{1}{2}(y^3-2y^2+y) - y^2 + y\right] dy \\
&= \int_0^1 (-y^3 + y + y\sqrt{1-y^2})dy \\
&= -\frac{y^4}{4} + \frac{y^2}{2} - \frac{1}{2}\cdot\frac{2}{3}(1-y^2)^{3/2}\Big]_0^1 \\
&= -\frac{1}{4} + \frac{1}{2} + \frac{1}{3} = \frac{7}{12}
\end{aligned}$$

4. **Problem:** Evaluate the double integral

$$\iint_R (x+\sqrt{y})dxdy$$

where R is the region bounded by $x=0$, $y=x^2$, $y=2-x$

5. **Problem:** Evaluate the integral $\iint_R x^2y dxdy$, where R is bounded by $x=0$, $x=1$, $y=\sqrt{x}$, and $y=2$.

6. **Problem:** Find the volume under the paraboloid $z=x^2+3y^2$ and above the region bounded by the lines $x=0$, $y=0$, and $x+y=1$.

7. **Problem:** Evaluate the iterated integral

$$\int_0^1 \int_x^1 \sin y^2 dy\, dx$$

8. **Problem** The rectangle R in the xy-plane consists of those points (x,y) for which $0 \leqq x \leqq 2$ and $0 \leqq y \leqq 1$. Find the volume V of the solid that lies below the surface $z = 1+xy$ and above R.

9. **Problem:** Compute by double integration the area A of the region R in the xy-plane that is bounded by the line $y=x$ and by the parabola $y=x^2-2x$.

10. **Problem:** Find the volume of the wedge-shaped solid T that lies above the xy-plane, below the plane $z = x$, and within the cylinder $x^2 + y^2 = 4$.

11. **Problem:** A lamina occupies the region bounded by the line $y = x + 2$ and by the parabola $y = x^2$. The density of the lamina at the point $P(x, y)$ is $\rho(x, y) = kx^2$ (where k is a positive constant). Find the mass and the center of mass of the lamina.

12. **Problem:** Evaluate the double integral $\iint_R (x - 3y^2)dxdy$, where $R = \{(x, y)|0 \le x \le 2,\ 1 \le y \le 2\}$.

13. **Problem:** Evaluate $\iint_R y \sin(xy)dxdy$, where $R = [1, 2] \times [0, \pi]$.

14. **Problem:** Find the volume of the solid S that is bounded by the elliptic paraboid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

15. **Problem:** If $R = [0, \pi/2] \times [0, \pi/2]$, compute $\iint_R \sin x \cos y dxdy$.

16. **Problem:** Evaluate $\iint_D (x+2y)dxdy$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

17. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

    If we integrate with respect to x first then we get,

$$
\begin{aligned}
V &= \iint_D (x^2 + y^2)dxdy = \int_0^4 \int_{y/2}^{\sqrt{y}} (x^2 + y^2)dx\ dy \\
&= \int_0^4 \left[\frac{x^3}{3} + y^2 x\right]_{x=y/2}^{x=\sqrt{y}} dy = \int_0^4 \left(\frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2}\right) dy \\
&= \frac{2}{15}y^{5/2} + \frac{2}{7}y^{7/2} - \frac{13}{96}y^4\Big]_0^4 = \frac{216}{35}
\end{aligned}
$$

18. **Problem:** Evaluate $\iint_D xy\ dxdy$ where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

19. **Problem:** Use the following change of variables $x = ar \cos \theta,\ y = br \sin \theta$ to evaluate

$$\iint_{D^*} e^{-\frac{x^2}{a^2} - \frac{y^2}{b^2}} dy\ dx,$$

where D^* is the quarter of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, \quad x \geq 0, \quad y \leq 0.$$

SOLUTIONS

Solution 1: An antiderivative of $1/(1+y^2)$ is $\tan^{-1} y$. So on integrating first with respect to y, holding x fixed, we get

$$\begin{aligned}\int_0^3 [x^2 \tan^{-1} y]_{-1}^1 dx &= \int_0^3 x^2[\tan^{-1} 1 - \tan^{-1}(-1)]dx \\ &= \int_0^3 x^2 \left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] dx \\ &= \frac{\pi}{2} \int_0^3 x^2 dx\end{aligned}$$

Now we integrate with respect to x to get

$$\frac{\pi}{2}\left[\frac{x^3}{3}\right]_0^3 = \frac{\pi}{2}\left(\frac{27}{3}\right) = \frac{9\pi}{2}$$

So we have the final result

$$\int_0^3 \int_{-1}^1 \frac{x^2}{1+y^2} dy\ dx = \frac{9\pi}{2}$$

Solution 2:

$$\begin{aligned}\iint_R (x+2y)dxdy &= \int_0^4 \int_0^1 (x+2y)dy\ dx \\ &= \int_0^4 \left[xy + y^2\right]_0^1 dx \\ &= \int_0^4 (x+1)dx \\ &= \left.\frac{(x+1)^2}{2}\right]_0^4 = \frac{25}{2} - \frac{1}{2} = \frac{24}{2} = 12\end{aligned}$$

We chose to integrate first with respect to y, then x. If we use the other order, we get

$$\begin{aligned}\iint_R (x+2y)dxdy &= \int_0^1 \int_0^4 (x+2y)dx\ dy \\ &= \int_0^1 \left[\frac{x^2}{2} + 2xy\right]_0^4 dy \\ &= \int_0^1 (8+8y)dy \\ &= \left. 8y + 4y^2\right]_0^1 = 8 + 4 = 12\end{aligned}$$

Solution 4:

$$\begin{aligned}
\iint_R (x+\sqrt{y})dxdy &= \int_0^1 \int_{x^2}^{2-x} (x+\sqrt{y})dy\ dx \\
&= \int_0^1 \left[xy + \frac{2}{3}y^{3/2}\right]_{x^2}^{2-x} dx \\
&= \int_0^1 \left[x(2-x) + \frac{2}{3}(2-x)^{3/2} - x^3 - \frac{2}{3}x^3\right] dx \\
&= \int_0^1 \left[2x - x^2 + \frac{2}{3}(2-x)^{2/3} - \frac{5}{3}x^3\right] dx \\
&= x^2 - \frac{x^3}{3} - \frac{2}{3}\cdot\frac{2}{5}(2-x)^{5/2} - \frac{5}{12}x^4\Big]_0^1 \\
&= 1 - \frac{1}{3} - \frac{4}{15} - \frac{5}{12} + \frac{4}{15}(2)^{5/2} = \frac{-1+64\sqrt{2}}{60} \approx 1.492
\end{aligned}$$

Solution 5:

$$\begin{aligned}
\iint_R x^2y dxdy &= \int_0^1 \int_{\sqrt{x}}^2 x^2y\ dy\ dx \\
&= \int_0^1 \left[\frac{x^2y^2}{2}\right]_{\sqrt{2}}^2 dx \\
&= \int_0^1 \left(2x^3 - \frac{x^3}{2}\right) dx = \frac{2x^3}{3} - \frac{x^4}{8}\Big]_0^1 = \frac{13}{24}
\end{aligned}$$

Solution 6:

$$\begin{aligned}
V = \iint_R (x^2+3y^2)dxdy &= \int_0^1 \int_0^{1-x} (x^2+3y^2)dy\ dx \\
&= \int_0^1 \left[x^2y + y^3\right]_0^{1-x} dx \\
&= \int_0^1 \left[x^2 - x^3 + (1-x)^3\right] dx \\
&= \frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{4}\Big]_0^1 \\
&= \left(\frac{1}{3} - \frac{1}{4}\right) + \frac{1}{4} = \frac{1}{3}
\end{aligned}$$

Solution 7:

$$\begin{aligned}
\int_0^1 \int_0^y \sin y^2 dx\ dy &= \int_0^1 \left[x\sin y^2\right]_0^y dy \\
&= \int_0^1 y\sin y^2 dy \\
&= -\frac{1}{2}\cos y^2\Big]_0^1 = \frac{1}{2}(1-\cos 1)
\end{aligned}$$

Solution 8:

$$\begin{aligned} V &= \int_R\int z dxdy = \int_0^2\int_0^1 (1+xy)dy\ dx \\ &= \int_0^2 \left[y + \frac{1}{2}xy^2\right]_{y=0}^1 dx = \int_0^2 (1+\frac{1}{2}x)dx = \left[x + \frac{1}{4}x^2\right]_0^2 = 3. \end{aligned}$$

Solution 9:

$$\begin{aligned} A &= \int_a^b\int_{\text{high[x]}}^{\text{low[x]}} 1\ dy\ dx = \int_0^3\int_{x^2-2x}^{x} 1\ dy\ dx \\ &= \int_0^3 [y]_{y=x^2-2x}^{x} dx = \int_0^3 (3x - x^2)dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3 = \frac{9}{2}. \end{aligned}$$

Solution 10:

$$\begin{aligned} V &= \int_S\int z\ dxdy = 2\int_0^2\int_0^{\sqrt{4-y^2}} x\ dx\ dy = 2\int_0^2 \left[\frac{1}{2}x^2\right]_{x=0}^{\sqrt{4-y^2}} dy \\ &= \int_0^2 (4-y^2)dy = \left[4y - \frac{1}{3}y^3\right]_0^2 = \frac{16}{3}. \end{aligned}$$

Solution 11: The line and the parabola intersect in the two points (-1,1) and (2,4).

$$\begin{aligned} m &= \int_{-1}^2\int_{x^2}^{x+2} kx^2\ dy\ dx = k\int_{-1}^2 \left[x^2y\right]_{y=x^2}^{x+2} dx \\ &= k\int_{-1}^2 (x^3 + 2x^2 - x^4)dx = \frac{63k}{20}. \\ \bar{x} &= \frac{20}{63k}\int_{-1}^2\int_{x^2}^{x+2} kx^3\ dy\ dx = \frac{20}{63}\int_{-1}^2 \left[x^3y\right]_{y=x^2}^{x+2} dx \\ &= \frac{20}{63}\int_{-1}^2 (x^4 + 2x^3 - x^5)dx = \frac{20}{63}\cdot\frac{18}{5} = \frac{8}{7}; \\ \bar{y} &= \frac{20}{63k}\int_{-1}^2\int_{x^2}^{x+2} kx^2y\ dy\ dx = \frac{20}{63}\int_{-1}^2 \left[\frac{1}{2}x^2y^2\right]_{y=x^2}^{x+2} dx \\ &= \frac{10}{63}\int_{-1}^2 (x^4 + 4x^3 + 4x^2 - x^4)dx = \frac{30}{7}. \end{aligned}$$

Solution 12:

$$\begin{aligned} \iint_R (x - 3y^2)dxdy &= \int_0^2\int_1^2 (x - 3y^2)dy\ dx \\ &= \int_0^2 \left[xy - y^3\right]_{y=1}^{y=2} dx \\ &= \int_0^2 (x - 7)dx = \left.\frac{x^2}{2} - 7x\right]_0^2 = -12 \end{aligned}$$

Integrating this time with respect to x first, we have

$$\iint_R (x-3y^2)dxdy = \int_1^2 \int_0^2 (x-3y^2)dx\ dy$$
$$= \int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_{x=0}^{x=2} dy$$
$$= \int_1^2 (2-6y^2)dy = 2y - 2y^3\Big]_1^2 = -12$$

Solution 13: If we first integrate with respect to x, we get

$$\iint_R y\sin(xy)dxdy = \int_0^\pi \int_1^2 y\sin(xy)dx\ dy$$
$$= \int_0^\pi [-\cos(xy)]_{x=1}^{x=2} dy$$
$$= \int_0^\pi (-\cos 2y + \cos y)dy$$
$$= -\frac{1}{2}\sin 2y + \sin y\Big]_0^\pi = 0$$

Solution 14:

$$V = \iint_R (16 - x^2 - 2y^2)dxdy = \int_0^2 \int_0^2 (16 - x^2 - 2y^2)dx\ dy$$
$$= \int_0^2 \left[16x - \frac{x^3}{3} - 2y^2x\right]_{x=0}^{x=2} dy$$
$$= \int_0^2 \left(\frac{88}{3} - 4y^2\right) dy = \left[\frac{88}{3}y - \frac{4}{3}y^3\right]_0^2 = 48$$

Solution 15:

$$\iint_R \sin x \cos y dxdy = \int_0^{\pi/2} \sin x\ dx \int_0^{\pi/2} \cos y\ dy$$
$$= [-\cos x]_0^{\pi/2} [\sin y]_0^{\pi/2}$$
$$= 1 \cdot 1 = 1$$

Solution 16: The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$.

$$\begin{aligned}\iint_D (x+2y)dxdy &= \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)dy\ dx \\ &= \int_{-1}^{1} \left[xy + y^2\right]_{y=2x^2}^{y=1+x^2} dx \\ &= \int_{-1}^{1} [x(1+x^2) + (1+x^2)^2 - x(2x^2) - (2x^2)^2]dx \\ &= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)dx \\ &= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\Big]_{-1}^{1} = \frac{32}{15}\end{aligned}$$

Solution 17:

$$\begin{aligned}V &= \iint_D (x^2+y^2)dxdy = \int_0^2\int_{x^2}^{2x} (x^2+y^2)dy\ dx \\ &= \int_0^2 \left[x^2y + \frac{y^3}{3}\right]_{y=x^2}^{y=2x} dx = \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2x^2 - \frac{(x^2)^3}{3}\right] dx \\ &= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3}\right) dx = -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6}\Big]_0^2 = \frac{216}{35}\end{aligned}$$

Solution 18:

$$\begin{aligned}\iint_D xy\ dxdy &= \int_{-2}^{4}\int_{y^2/2-3}^{y+1} xy\ dx\ dy = \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{x=y^2/2-3}^{x=y+1} dy \\ &= \frac{1}{2}\int_{-2}^{4} y\left[(y+1)^2 - \left(\frac{y^2}{2} - 3\right)^2\right] dy \\ &= \frac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy \\ &= \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + 2\frac{y^3}{3} - 4y^2\right]_{-2}^{4} = 36\end{aligned}$$

Solution 19 Let us find the jacobian determinant of the given change of variables: $x = ar\cos\theta,\ y = br\sin\theta,\ x \geq 0,\ y \leq 0$ as

$$\text{jacobiandet}[\{ar\cos\theta, br\sin\theta\}, \{r,\theta\}] = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr.$$

Since the boundary of the region D^* in the xy-plane is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x \geq 0, y \leq 0,$$

the $r\theta$-equations for boundaries of the region D in the $r\theta$-plane are the following

$$r = 1, \quad \frac{3\pi}{2} \leq \theta \leq 2\pi.$$

Therefore the double integral can be computed as

$$\iint_{D^*} e^{-\frac{x^2}{a^2} - \frac{y^2}{b^2}} dy\, dx = ab \int_{\frac{3\pi}{2}}^{2\pi} \left(\int_0^1 re^{-r^2} dr \right) d\theta$$

$$= -\frac{ab}{2} \int_{\frac{3\pi}{2}}^{2\pi} \left(e^{-r^2} |_0^1 \right) d\theta$$

$$= -\frac{ab}{2} \left(\frac{1}{e} - 1 \right) \int_{\frac{3\pi}{2}}^{2\pi} d\theta = \left(1 - \frac{1}{e} \right) \frac{\pi ab}{4}.$$