

PracticeMe7. Do these Problems Before HW7
If you discover any misprints, please tell your Instructor

PROBLEMS

Problem 1. Let S be the cone $z = \sqrt{x^2 + y^2}$; $0 \leq z \leq 1$. Compute $\iint_S x^2 dS$.

Problem 2. Let S be the hemisphere $x^2 + y^2 + z^2 = 1$ with $z \geq 0$. Evaluate $\iint_S (x^2 + y^2 - 1) dS$.

Problem 3. Evaluate $\iint_S (x^2 + y^2) dS$ where S is the portion of the plane $x + y + z = 10$ that is inside the cylinder $x^2 + y^2 = 4$.

Problem 4. Evaluate $\iint_S \sqrt{x} dS$ where S is the surface defined by $P(u, v) = (u^2, v + u, v)$ $0 \leq u \leq 1, 0 \leq v \leq 1$.

In problems 5-7, apply the Divergence theorem to evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ (the flow of the vector field across the surface) for the given choice of F and the boundary surface of S . For each closed surface, assume \mathbf{n} is the outward unit normal vector.

Problem 5. $F = \{x, y, z\}$ and S is the sphere $x^2 + y^2 + z^2 = a^2$.

Problem 6. $F = \{xy, yz, xz\}$; S is the cube $0 \leq x \leq 1; 0 \leq y \leq 1; 0 \leq z \leq 1$.

Problem 7. $F = \{e^{z^2} \cos(z), x^3 e^{z^3}, 3z\}$ S is the surface bounded by the cylinder $x^2 + y^2 = 16$ and the planes $z = 0$ and $z = 7$.

In problems 8-9, use Stokes' Theorem to evaluate $\iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S}$.

Problem 8. $F = \{z, y, e^{z^2} x\}$; S is the surface $z = 4 - x^2 - y^2$; $0 \leq z \leq 4$

Problem 9. $F = \{xy, xz, x\}$; S is the surface consisting of the all faces of the cube $0 \leq x \leq 1; 0 \leq y \leq 1; 0 \leq z \leq 1$ except the face lying in the xy -plane.

In problems 10-11, use Stokes' Theorem to evaluate the circulation of F along C . In each case C is oriented counterclockwise as viewed from above.

Problem 10. $F = \{xz, 2xy, 3xy\}$. C is the boundary of the part of the plane $3x + y + z = 3$ in the first octant.

Problem 11. $F = \{x, y, -10x\}$; C is the curve intersection of the plane $z = 5 - x - y$ and the cylinder $x^2 + y^2 = 1$.

SOLUTIONS

Solution 1. Here is a parametrization of this cone.

$$P(r, t) = \{r \cos(t), r \sin(t), r\} \quad 0 \leq r \leq 1, \quad 0 \leq t \leq 2\pi.$$

Notice $z = \sqrt{r^2 \cos^2(t) + r^2 \sin^2(t)} = \sqrt{r^2} = r$.

$$\begin{aligned} S(r, t) &= SA[P[r, t], \{r, t\}] = \left\| \frac{\partial P}{\partial r} \times \frac{\partial P}{\partial t} \right\| \\ \frac{\partial P}{\partial r} &= \{\cos(t), \sin(t), 1\} \\ \frac{\partial P}{\partial t} &= \{-r \sin(t), r \cos(t), 0\} \\ \frac{\partial P}{\partial r} \times \frac{\partial P}{\partial t} &= \{-r \cos(t), -r \sin(t), r\} \\ \left\| \frac{\partial P}{\partial r} \times \frac{\partial P}{\partial t} \right\| &= \sqrt{r^2 + r^2} = r\sqrt{2}. \end{aligned}$$

$$\begin{aligned} \iint_S x^2 dS &= \int_0^{2\pi} \int_0^1 r^2 \cos^2(t) S(r, t) dr dt \\ &= \sqrt{2} \int_0^{2\pi} \int_0^1 r^3 \cos^2(t) dr dt \\ &= \frac{\sqrt{2}}{4} \int_0^{2\pi} \cos^2(t) dt = \frac{\sqrt{2}}{4} \left[\frac{t}{2} + \frac{1}{4} \sin[2t] \right]_0^{2\pi} \\ &= \frac{\pi\sqrt{2}}{4}. \end{aligned}$$

Solution 2. A parametrization of S is given in spherical coordinates by

$$\begin{aligned} P[s, t] &= \{\sin(s) \cos(t), \sin(s) \sin(t), \cos(s)\} \quad 0 \leq s \leq \pi/2; \quad 0 \leq t \leq 2\pi \\ \left\| \frac{\partial P}{\partial s} \times \frac{\partial P}{\partial t} \right\| &= \sin(s). \end{aligned}$$

$$\begin{aligned} \iint_S (x^2 + y^2 - 1) dS &= \iint_S -z^2 dS \\ &= \int_0^{\pi/2} \int_0^{2\pi} -\cos^2(s) \sin(s) dt ds \\ &= 2\pi \int_0^{\pi/2} -\cos^2(s) \sin(s) ds \\ &= 2\pi \left[\frac{\cos^3(s)}{3} \right]_0^{\pi/2} = -\frac{2\pi}{3} \end{aligned}$$

Solution 3. Let us parametrize this surface S .

$$P[r, t] = \{r \cos(t), r \sin(t), 10 - r \cos(t) - r \sin(t)\}, \quad 0 \leq r \leq 2; \quad 0 \leq t \leq 2\pi.$$

$$\begin{aligned} \frac{\partial P}{\partial r} &= \{\cos(t), \sin(t), -\cos(t) - \sin(t)\} \\ \frac{\partial P}{\partial t} &= \{-r \sin(t), r \cos(t), -r \cos(t) + r \sin(t)\} \\ \frac{\partial P}{\partial r} \times \frac{\partial P}{\partial t} &= \{r, r, r\} \\ \left\| \frac{\partial P}{\partial r} \times \frac{\partial P}{\partial t} \right\| &= r\sqrt{3} \\ \iint_S (x^2 + y^2) dS &= \int_0^2 \int_0^{2\pi} r^2 \sqrt{3} r \, dt \, dr \\ &= 2\pi\sqrt{3} \int_0^2 r^3 \, dr = 2\pi\sqrt{3} \left[\frac{r^4}{4} \right]_0^2 = 8\pi\sqrt{3} \end{aligned}$$

Solution 4.

$$\begin{aligned} \frac{\partial P}{\partial u} &= \{2u, 1, 0\} \\ \frac{\partial P}{\partial v} &= \{0, 1, 1\} \\ \frac{\partial P}{\partial u} \times \frac{\partial P}{\partial v} &= \begin{vmatrix} i & j & k \\ 2u & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \{1, -2u, 2u\} \\ \left\| \frac{\partial P}{\partial u} \times \frac{\partial P}{\partial v} \right\| &= \sqrt{1 + 8u^2} \\ \iint_S \sqrt{x} dS &= \int_0^1 \int_0^1 u \sqrt{1 + 8u^2} \, dv \, du \\ &= \int_0^1 u \sqrt{1 + 8u^2} \, du \\ &= \frac{2}{3} \times \frac{1}{16} (1 + 8u^2)^{3/2} \Big|_0^1 = \frac{13}{12} \end{aligned}$$

Solution 5.

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_V \operatorname{div}(F) \, dx \, dy \, dz \\ &= \iiint_V 3 \, dx \, dy \, dz = 3(\text{volume of the sphere}) \\ &= 3 \left(\frac{4\pi a^3}{3} \right) = 4\pi a^3. \end{aligned}$$

Solution 6.

$$\begin{aligned}
\text{flow across} &= \iint_S \mathbf{F} \cdot d\mathbf{S} = \\
&= \iiint_V \operatorname{div}(F) dx dy dz \\
&= \int_0^1 \int_0^1 \int_0^1 (x + y + z) dx dy dz = \frac{3}{2}
\end{aligned}$$

Solution 7.

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_V \operatorname{div}(F) dx dy dz \\
&= \iiint_V 3 dx dy dz = 3(\text{volume}) \\
&= 3(\pi 4^2) \cdot 7 = 336\pi
\end{aligned}$$

Solution 8. The boundary of that surface is the circle C having the equation $x^2 + y^2 = 4$ in the xy -plane. We parametrize the boundary of this circle by $x = 2 \cos(t)$; $y = 2 \sin(t)$, $z = 0$.

$$\begin{aligned}
I &= \oint_C z dx + y dy + e^{z^2} x dz \\
&= \oint_C 0 dx + y dy + e^{z^2} x 0 = \oint_C y dy \\
&= \int_0^{2\pi} 2 \sin(t) 2 \cos(t) dt = 2 \int_0^{2\pi} \sin(2t) dt \\
&= -\cos(2t) \Big|_0^{2\pi} = 0.
\end{aligned}$$

Solution 9. The boundary of this surface is the square C ; $0 \leq x \leq 1$; $0 \leq y \leq 1$ in the xy -plane.

$$\begin{aligned}
I &= \oint_C xy dx + xz dy + x dz \\
&= \oint_C xy dx + 0 dy + (x)0 \\
&= \oint_C xy dx = \int_{C_1} xy dx + \int_{C_2} xy dx + \int_{C_3} xy dx + \int_{C_4} xy dx
\end{aligned}$$

where C_1, C_2, C_3, C_4 are the four sides of the square.

On C_1 , $y = 0$ and on C_4 , $x = 0$.

On C_2 $dx = 0$ and on C_3 , $y = 1$.

So

$$I = \oint_{C_3} xy dx = \oint_{C_3} x dx = \int_1^0 x dx = \frac{x^2}{2} \Big|_1^0 = -\frac{1}{2}$$

Solution 10. A parametrization of surface (triangle) inside C in the plane $3x + y + z = 3$ can be obtained as $\{x, y, g(x, y)\}$ where $g(x, y) = 3 - 3x - y$. $0 \leq x \leq 1$; $0 \leq y \leq 3 - 3x$.

The outward normal is $\{3, 1, 1\}$.

$$F(x, y, g(x, y)) = \{x(3 - 3x - y), 2xy, 3xy\}$$

$$\text{Curl}F(x, y, z) = \{3x, -3y + x, 2y\}.$$

$$\text{Hence } \text{Curl}F(x, y, g(x, y)) = \{3x, -3y + x, 2y\}.$$

The flow along C is equal to the flow across the surface of $\text{Curl}F$ which is:

$$\begin{aligned} & \int_0^1 \int_0^{3-3x} \text{curl}F(x, y, g(x, y)) \cdot \text{normal}(x, y) dy dx \\ & \int_0^1 \int_0^{3-3x} \{3x, -3y + x, 2y\} \cdot \{3, 1, 1\} dy dx \\ & = \int_0^1 \int_0^{3-3x} (10x - y) dy dx = \frac{7}{2} \end{aligned}$$

Solution 11. A parametrization of the surface inside the curve is given by.

$$P[r, t] = \{r \cos(t), r \sin(t), 5 - r \cos(t) - r \sin(t)\} \quad 0 \leq r \leq 1; \quad 0 \leq t \leq 2\pi$$

$$\text{The outward normal } (r, t) = \frac{\partial P}{\partial r} \times \frac{\partial P}{\partial t} = \{r, r, r\}.$$

$$\text{The curl of } F \text{ is } = \{0, 10, 0\}.$$

$$\text{Curl}F \cdot \text{normal } (r, t) = \{0, 10, 0\} \cdot \{r, r, r\} = 10r$$

The flow along C is equal to the flow across the surface of $\text{Curl}(F)$ which is

$$\begin{aligned} & \int_0^1 \int_0^{2\pi} \text{Curl}F \cdot \text{normal}(r, t) dt dr \\ & \int_0^1 \int_0^{2\pi} 10r dt dr = 20\pi \int_0^1 r dr = 10\pi. \end{aligned}$$