Qualifying Examination in Analysis

January, 2008

Solutions

Real Analysis I: One-dimensional Calculus

1. **(a)** For each fixed x, $(x-n)^2 \to \infty$, as $n \to \infty$, hence

$$|f_n(x)| \leq \frac{1}{1+(x-n)^2} \to 0.$$

(b) From (a), if f_n converges uniformly, it converges uniformly to 0, hence $\sup_{\mathbb{R}} |f_n(x)| \to 0$, as $n \to \infty$. However

$$|f_n(n)| = \cos(1) \leq \sup_{\mathbb{R}} |f_n(x)| \to 0$$

which is a contradiction.

If $x \leq A$, then for $n > A$ we have $(x-n)^2 \geq (n-A)^2$. Hence $|f_n(x)| \leq \dfrac{1}{1+(A-n)^2} \to 0$. This means that $\sup_{x \leq A} |f_n(x)| \to 0$, hence convergence is uniform on $(-\infty, A]$.

(c) Use M-test, since $\displaystyle\sum_{n>A} \frac{1}{1+(A-n)^2} < \infty$.

2. **(a)** Let $H(x) = f(x)G(x) - g(x)F(x)$. From the Fundamental Theorem of Calculus, since f, g are continuous then $F' = f$, and $G' = g$. Hence, $H'(x) = f'(x)G(x) - g'(x)F(x) \geq 0$ and $H(0) = 0$, so that H is increasing on $[0, \infty)$, which implies $H(x) \geq H(0) = 0$, for $x \geq 0$.

$$\frac{d}{dx}\left(\frac{F(x)}{G(x)}\right) = \frac{f(x)G(x) - g(x)F(x)}{G(x)^2} \geq 0.$$

(b) If $f(x_0) > 0$ then, since f increasing, then $f(t) \geq f(x_0) > 0$, for any $t \geq x_0$. Hence, as $x \to +\infty$

$$F(x) = \int_0^{x_0} f(t)dt + \int_{x_0}^{x} f(t)dt \geq F(x_0) + f(x_0)(x - x_0) \to +\infty.$$

Similarly, $g(x) \geq L > 0$, hence $G(x) \geq Lx \to +\infty$.

(c) In the given hypothesis l'Hospital rule applies, and hence

$$\lim_{x\to\infty} \frac{F(x)}{G(x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)}.$$

Real Analysis II: Multi-dimensional Calculus

1. **(a)** We have

$$\frac{\partial f}{\partial x} = 2x + z, \quad \frac{\partial f}{\partial y} = 3y^2, \quad \frac{\partial f}{\partial z} = 1 + x + 3z^2$$

which are continuous on $\mathbb{R}^2$. Since f is C^1, and $f(0,0,0) = 0$ and $\frac{\partial f}{\partial z}(0,0,0) = 1$, the implicit function theorem applies.

(b) Differentiating $f(x, y, g(x, y)) = 0$ with respect to x we obtain

$$\frac{\partial f}{\partial x}(x, y, g(x, y)) + \frac{\partial f}{\partial z}(x, y, g(x, y))\frac{\partial g}{\partial x}(x, y) = 0. \qquad (1)$$

Hence, since $g(0,0) = 0$ then $\frac{\partial g}{\partial x}(0,0) = 0$. Similarly one finds $\frac{\partial g}{\partial y}(0,0) = 0$.

Differentiating in x the above equation (1) yields

$$\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial z \partial x}\frac{\partial g}{\partial x} + \frac{\partial^2 f}{\partial z^2}\left(\frac{\partial g}{\partial x}\right)^2 + \frac{\partial f}{\partial z}\frac{\partial^2 g}{\partial x^2}$$

and evaluating at the origin gives $\frac{\partial^2 g}{\partial x^2}(0,0) = -2$.

(c) We have $f(0, y, z) = 0$, with $z = g(0, y)$. Hence $z^3 + z = -y^3$, and this means (since $z^3 + z = z(z^2 + 1)$) that $g(0, y)$ changes sign around $y = 0$.

2. **(a)** We have $f(0, y) = f(x, 0) = 0$ hence both partials are 0 at the origin.

(b) Since both partials at the origin are 0, to check differentiability all we need to do is check whether

$$\lim_{(x,y)\to(0,0)} \frac{f(x, y)}{(x^2 + y^2)^{1/2}} = 0$$

But this follows from

$$\left|\frac{f(x,y)}{(x^2+y^2)^{1/2}}\right| \le \frac{|x^2y^3|}{(x^2+y^2)^{1/2}} \le \frac{|x^2y^3|}{(y^2)^{1/2}} = x^2y^2 \to 0.$$

(b) We have

$$\frac{\partial f}{\partial x} = 2xy^3 \cos\frac{1}{x^4+y^4} - \frac{4x^5y^3}{(x^4+y^4)^2}\sin\frac{1}{x^4+y^4}.$$

The first term converges to 0, as $(x,y) \to (0,0)$, whereas the second term does not have limit, for example along the line $x = y$.

Complex Analysis

1. **(a)**

$$\begin{aligned}
&\int_\gamma e^{x-iy}\,dz \\
&= \int_{-1}^{1} ie^{1-iy}\,dy + \int_{1}^{-1} e^{x-i}\,dx + \int_{1}^{-1} ie^{-1-iy}\,dy + \int_{-1}^{1} e^{x+i}\,dx \\
&= i(e-e^{-1})\int_{-1}^{1} e^{-iy}\,dy + (e^i - e^{-i})\int_{-1}^{1} e^x\,dx \\
&= 2(e-e^{-1})(e^i-e^{-i}) = 8i\sinh 1 \sin 1.
\end{aligned}$$

Note: the function $e^{\bar{z}}$ is not holomorphic inside the square.

(b) Let $\Gamma = \gamma \cup [2,0]$; this simple closed curve contains $z = -i$ and does not contain $z = i$. Then $\int_\Gamma \frac{dz}{1+z^2}$ is $2\pi i$ times the residue of $1/(1+z^2)$ at $z = -i$, that is

$$\int_\Gamma \frac{dz}{1+z^2} = -\pi.$$

But

$$\int_{[2,0]} \frac{dz}{1+z^2} = -\tan^{-1}(2).$$

Hence

$$\int_\gamma \frac{dz}{1+z^2} = -\pi + \tan^{-1}(2).$$

2. **(a)** If f is analytic on a small punctured disk $\{0 < |z| < \epsilon\}$ then in that domain f has a Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$. The origin is a removable singularity if $a_n = 0$ for all $n < 0$; it is a pole of order $m > 0$ if $a_n = 0$ for all $n < m$ and $a_m \neq 0$; it is an essential singularity otherwise (i.e. $a_n \neq 0$ for infinitely many $n < 0$).

   (b) If f has a pole or a removable singularity at 0 then $\lim_{z\to 0} |f(z)|$ either is finite or infinite. But along $z = x$ real, the function $\left|\sin \frac{1}{\sin x}\right|$ does not have a limit as $x \to 0$, so the origin is an essential singularity.

   Next, $\sin z = zh(z)$ and $e^z - 1 = zk(z)$, where h and k are entire functions with $h(0), k(0) \neq 0$. Then $g(z) = z^{-4}k(z)/h(z)$ has a pole of order 4 at $z = 0$.

   (c) $f(z) = az^{-1} + h(z)$, where h is entire, and $a \neq 0$. Hence $e^{f(z)} = e^{a/z}e^{h(z)}$. But $e^{h(z)}$ is entire, whereas $e^{a/z}$ has an essential singularity (e.g. look at its Laurent expansion). Thus 0 is an essential singularity, since e.g. $\lim_{z\to 0} |f(z)|$ is not finite or infinite.