Qual. Exam - Jan 07 - Solutions (or Hints)

I-1 a) $\{a_n\}$ is Cauchy Convergent iff $\forall \varepsilon > 0 \ \exists N : |a_n - a_m| < \varepsilon \ \forall m, n \geq N$

b) $\sum a_n$ converges iff $S_n := \sum_{k=1}^{n} a_k$ converges to a finite limit as $n \to +\infty$

c) If $S_n = \sum_{1}^{n} a_k$, $t_n = \sum_{k=1}^{n} b_k$ then $\forall \varepsilon > 0 \ \exists N : |t_m - t_n| < \varepsilon \ \forall m > n \geq N$

$$\Rightarrow |S_m - S_n| = \sum_{k=n+1}^{m} a_k \leq \sum_{k=n+1}^{m} b_k = |t_m - t_n| < \varepsilon \quad \forall m > n \geq N$$

$\leadsto \{S_n\}$ converges.

II-2 a) From the Fund. thm. of Calculus $\exists F'(x) = 2f(x) - 2f(x)f'(x) =$
$= 2f(x)(1 - f'(x)) \geq 0 \quad x > 0$

Hence F is increasing on $(0, +\infty)$ and on $[0, +\infty)$ (F cont.)

b) $F(0) = 0 \Rightarrow F(x) \geq F(0) = 0$

c) $G(x) := \left(\int_0^x f\right)^2 - \int_0^x f^3 \Rightarrow G'(x) = 2f(x)\int_0^x f - f^3(x)$

$$= f(x)\left[2\int_0^x f - f^2(x)\right] \geq 0.$$

II-1 a) By the Implicit Function Thm: all the hypothesis are satisfied!

b) $f(\gamma(y), y) = 0$ by I.F.T γ' exists, γ'' exists

by chain rule $\gamma'(y) \frac{\partial f}{\partial x}(\gamma(y), y) + \frac{\partial f}{\partial y}(\gamma(y), y) = 0$

$y = 0 \Rightarrow \gamma'(0) = 0$

e) $G(y) = \frac{\partial f}{\partial y}(\gamma(y), y) \Rightarrow G'(0) = \gamma'(0)\frac{\partial^2 f}{\partial x \partial y}(0,0) + \frac{\partial^2 f}{\partial y^2}(0,0) \neq 0$

Since $G(0) = 0$ G must change sign around $y = 0$.

(increasing or decreasing strictly around $y = 0$)

II-2 a) $\frac{\partial f}{\partial x} = -\sin x \cos y$ $\quad \frac{\partial^2 f}{\partial x^2} = -\cos x \cos y$

$\frac{\partial^2 f}{\partial x \partial y} = \sin x \sin y$ $\quad \frac{\partial f}{\partial y} = -\sin y \cos x$ $\quad \frac{\partial^2 f}{\partial y^2} = -\cos y \cos x$

$|d^2 f((a,b),(h,k))| \leq h^2 + 2hk + k^2 = (h+k)^2$ $\quad h,k \geq 0$, $a,b \in \mathbb{R}$

b) By Mean V. Thm.

$f(x,y) = f(0,0) + x\frac{\partial f}{\partial x}(0,0) + y\frac{\partial f}{\partial y}(0,0) + \frac{1}{2}d^2 f((a,b),(x,y))$

$= 1 + \frac{1}{2}d^2 f((a,b),(h,k))$ $\quad$ some (a,b)

$\Rightarrow |f(x,y) - 1| \leq \frac{1}{2}(x+y)^2$ $\quad$ on A

c) Area of $A = \frac{\pi}{4}$

d) $\left|\int_A f(x,y)\,dx\,dy - \int_A 1\,dx\,dy\right| \leq \int_A |f(x,y) - 1|\,dx\,dy$

$\leq \frac{1}{2}\iint_{\substack{x^2+y^2 \leq 1 \\ x \geq 0 \\ y \geq 0}} (x+y)^2\,dx\,dy = \frac{1}{2}\int_0^1\int_0^{\pi/2} r(r^2 + r^2 \sin 2\theta)\,d\theta\,dr$

$= \frac{1}{2}\int_0^1 r^3\,d\theta \int_0^{\pi/2}(1 + \sin 2\theta)\,d\theta = \frac{1}{8}\left[-\frac{1}{2}\cos 2\theta + \theta\right]_0^{\pi/2} =$

$= \frac{1}{8}\left(1 + \frac{\pi}{2}\right) = \frac{\pi + 2}{16}$.

III-1 a) $h(z) = \frac{f(z)}{g(z)}$ is entire and bounded, hence constant.

b) $h(z)$ has a removable singularity at $z=0$, since the singularity is isolated and $|h(z)| < 1$ if $z \neq 0$.
Extending h at 0 gives an entire + bounded function
$\Rightarrow$ constant.

c) By the identity principle $g \equiv 0$ hence $f \equiv 0 \Rightarrow f = cg$ on $\mathbb{C}$ $(c=1)$.

III-2 a) Let D be a domain in $\mathbb{C}$, $\gamma \subset D$ a simple closed curve, $I(\gamma)$ = interior of γ. If f holomorphic in D except finitely many poles inside D ..
then

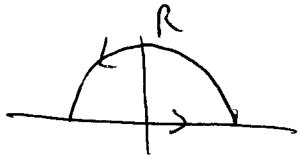
$$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{res}(f, z_j)$$

$z_1 \ldots z_n$ = poles inside γ (i.e. in $I(\gamma)$), $\operatorname{res}(f, z_j)$ = residue of f at z_j

b) $$f(z) = \frac{e^{iz}}{(z-1)^2+4} = \frac{e^{iz}}{(z-1-2i)(z-1+2i)}$$

$$\operatorname{res}(f, 1+2i) = \lim_{z \to 1+2i} f(z)(z-1-2i) = \frac{e^{(1+2i)i}}{4i} = \frac{e^{-2}e^{i}}{4i}$$

$$\operatorname{res}(f, 1-2i) = \frac{e^{(1-2i)i}}{-4i} = -\frac{e^{2}e^{i}}{4i}$$

c) $\gamma_R =$ 

$R>10$ so $1+2i \in I(\gamma_R)$

$$\int_{\gamma_R} f = 2\pi i \left(\frac{e^{-2}e^{i}}{4i}\right) = \frac{\pi}{2} e^{-2} e^{i}$$

$$\int_{\gamma_R} f = \int_{-R}^{R} \frac{e^{ix}}{(x-1)^2+4}\,dx + R\int_0^{\pi} e^{i\theta}\frac{e^{iRe^{i\theta}}}{(Re^{i\theta}-1)^2+4}\,d\theta$$

$$\left| R\int_0^{\pi} e^{i\theta}\frac{e^{iR(\cos\theta+i\sin\theta)}}{(Re^{i\theta}-1)^2+4}\,d\theta\right| \leq R\int_0^{\pi}\frac{e^{-R\sin\theta}}{(R-1)^2-4}\,d\theta \leq \frac{\pi R}{(R-1)^2-4} \to 0$$

$R \to +\infty$

[use: $\left|(Re^{i\theta}-1)^2+4\right| \geq \left|(Re^{i\theta}-1)^2\right| - 4 \geq (R-1)^2-4$

Since $\mathrm{Re}\int_{\gamma_R} f = \frac{\pi}{2e^2}\cos 1 = \int_{-R}^{R}\frac{\cos x}{(x-1)^2+4}\,dx + (\text{something} \to 0 \text{ as } R\to\infty)$

then $\lim_{R\to\infty}\int_{-R}^{R}\frac{\cos x}{(x-1)^2+4}\,dx$ exists finite and equals $\frac{\pi}{2e^2}\cos 1$

[you don't need to show that $\int_{-\infty}^{\infty}\frac{\cos x}{(x-1)^2+4}\,dx$ exists

__before__ applying the above method! it comes with it...]