Qualifying Examination

(*August 2003*)

Algebra

1. (i) (3 points) Let $\mathbb{Z}$ be the ring of integers. Find the number of elements in the cyclic subgroup of $\mathbb{Z}_{42}$ generated by 30;
(ii) (3 points) Consider the following permutation in the symmetric group S_{10}:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 10 & 9 & 1 & 2 & 6 & 4 & 5 & 3 & 8 \end{pmatrix}.$$

Write σ as a product of transpositions;
(iii) (4 points) Let G be a group. Prove that if there exists a unique element a of order 2 in G, then $ax = xa$ for all $x \in G$.

2. (i) (3 points) State the Fundamental Theorem for finitely generated abelian groups;
(ii) (3 points) List all abelian groups, up to isomorphisms, of order 300.
(iii) (4 points) Show that a finite abelian group is not cyclic if and only if it contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

3. (5 points) Classify all ring homomorphisms from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$.

4. (5 points) Determine if the following polynomials are irrducible over the field F indicated (justify your answer):
(i) $x^3 - 9$ over $F = \mathbb{Z}_{13}$;
(ii) $x^4 + 2x^2 + 2$ over $F = \mathbb{Q}$.

Linear Algebra

1. (5 points) Prove that every $m \times n$ matrix A of rank 1 is of the form $A = XY^t$, where X, Y are m- and n-dimensional column vectors.

2. (5 points) Find all (not necessarily invertible) linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ which carry the line $y = x$ to the line $y = 3x$.

3. (5 points) Compute $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{2003}$.

4. (5 points) The vector space $\mathcal{P}$ of all real polynomials of degree ≤ 3 has a bilinear form, defined by

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\mathrm{d}x.$$

Find an orthonormal basis for $\mathcal{P}$.

5. (5 points) Let V be the vector space of real $n \times n$ matrices. Consider the bilinear form on V defined by:

$$\langle A, B \rangle = \mathrm{trace}(A^t B).$$

Is it true that $\langle A, B \rangle > 0$ whenever $A \neq 0$? Justify your answer.

6. (5 points) A linear transformation T on a vector space V is called *nilpotent* if $T^k = 0$ for some positive integer k. Let T be a nilpotent linear transformation on a finite dimensional vector space V, and let W^i be the image of T^i. Prove that if $W^i \neq 0$, then

$$\dim W^{i+1} < \dim W^i.$$