Qualifying Examination

(*January 2006*)

- If you have any difficulty with the wording of the following problems please contact the supervisor immediately. All persons responsible for these problems, in principle, will be accessible during the entire duration of the exam.
- You are allowed to rely on a previous part of a multi-part problem even if you do not prove the previous part.
- Notations: $\mathbb{N}$ denotes the positive integers, $\mathbb{Z}$ the integers,

  $$\mathbb{Z}_n = \{0, 1, 2, \cdots n - 1\}$$

  the group of integers modulo n, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers, and $\mathbb{C}$ the complex numbers, S_n the symmetric group on n letters, A_n the alternating group on n letters, D_n the dihedral group with **2n** elements.

Algebra

1. Notation: If A and B are two subgroups of a group G, then

$$AB = \{ab | a \in A, b \in B\}.$$

(a) (3 points) Prove that AB is a subgroup if and only if $AB = BA$.

(b) (4 points) List up to isomorphism all abelian groups of order

$$108 = 2^2 3^3.$$

How many non isomorphic abelian groups of order 108 are there? How many of these are cyclic?

(c) (2 points) Prove that the center of a group G is a subgroup of G and is a normal subgroup of G.

(d) (2 points) What is the smallest n such that A_n has an element of order 6? Why?

2. State True or False. If true, prove it. If false, give an example to disprove it. You cannot just state a theorem to disprove anything.

(a) (3 points) There are no fields of order 32.

(b) (3 points) There is no nontrivial group homomorphism $\phi : \mathbb{Z}_5 \to S_3$.

(c) (3 points) $S_3 \times \mathbb{Z}_4$ is isomorphic to D_{12}.

3.

(a) (4 points) Determine (with justifications) if the following are irreducible in $\mathbb{Q}[x]$.

(i) $x^6 - 3x^4 + 12x^2 - 9x + 6$;

(ii) $x^3 + 6x^2 + 5x + 25$.

(b) (3 points) R is a commutative ring, and I and J are two ideals in R. P is a prime ideal of R such that $I \cap J \subset P$. Show that either $I \subset P$ or $J \subset P$.

(c) (3 points) R is a ring. $a, b \in R$ are two elements of R. Show that $1 - ab$ is invertible if and only if $1 - ba$ is invertible.

Linear Algebra

1. Let V be a finite dimensional vector space and $T : V \to V$ a linear transformation.

(i) [3 pts] Show that T is invertible if and only if the minimal polynomial of T has a nonzero constant term.

(ii) [2 pts] Show that if T is invertible, then T^{-1} can be expressed as a polynomial in T.

2. [5 pts] Find all possible Jordan canonical forms for a 5×5 complex matrix whose minimal polynomial is $(x-2)^2(x-1)$.

3. [5 pts] A matrix N is said to be *nilpotent* if $N^r = O$ for some integer $r \geq 1$. Show that, if N is an $n \times n$ nilpotent matrix N, then $I + N$ is invertible, where I is the identity matrix.

4. [5 pts] Let M be an $n \times n$ matrix over a field $\mathbb{F}$, and suppose that $\operatorname{tr}(MX) = 0$ for all $n \times n$ matrices X with entries in $\mathbb{F}$. Show that $M = O$.

5. [5 pts] Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix over the field of complex numbers. Find necessary and sufficient conditions on a, b, c, d to insure that A has two linearly independent eigenvectors.

6. [5 pts] Prove that any real $n \times n$ matrix M can be written in form

$$M = A + S + cI$$

where A is skew-symmetric (i.e., $A^t = -A$), S is symmetric (i.e., $S^t = S$), c is a scalar, I is the identity matrix, and $\operatorname{tr}(S) = 0$.