Qualifying Examination

(*August 2007*)

- If you have any difficulty with the wording of the following problems please contact the supervisor immediately. All persons responsible for these problems, in principle, will be accessible during the entire duration of the exam.

- You are allowed to rely on a previous part of a multi-part problem even if you do not prove the previous part.

- $\mathbb{Z}$ denotes the group (or ring) of integers with the usual operations, and $\mathbb{Z}_n$ denotes the quotient group (or ring) $\mathbb{Z}/n\mathbb{Z}$.

- $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the groups (or rings, or fields) of rational, real and complex numbers respectively with the usual operations.

- $\mathbb{C}^* = \mathbb{C} - \{0\}$ is the group with the usual multiplication.

- All matrices are over fields. For a field K, $K^{m\times n}$ denotes the set of all $m\times n$ matrices with entries in K.

Abstract Algebra

Each part (a), (b) or (c) below is worth 3 *points, and the total points are* 30.

1. (a) Does $[2007] \in \mathbb{Z}_{123456}$ generate the group $\mathbb{Z}_{123456}$? Justify your answer.

(b) Let H be a subgroup of a finite group G. Let $a \in G$, and n_H be the least positive integer n such that $a^n \in H$. Prove that n_H divides the order of a.

(c) Let $S^1 = \{z \in \mathbb{C}^* || z| = 1\}$ be the unit circle considered as a multiplicative subgroup of $\mathbb{C}^*$. Prove that

$$\mathbb{R}/\mathbb{Z} \cong S^1.$$

2. (a) Let I be an ideal of a commutative ring R. Define

$$\sqrt{I} = \{r \in R |\, r^n \in I \text{ for some positive integer } n\}.$$

Prove that $\sqrt{I}$ is an ideal of R.

(b) Prove or give a counter-example to the following statement:
Let A be a subring of a ring R. If P is a prime ideal of R, then $P \cap A$ is a prime ideal of A.

3. (a) Let $i = \sqrt{-1}$. Find all the automorphisms of the field $\mathbb{Q}[i]$. Justify your answer.

(b) Let $\mathbb{C}[x, y, z, w]$ be the polynomial ring with variables x, y, z, w. Let

$$I = \big\{f \in \mathbb{C}[x, y, z, w] |\; f(1, 2, 3, 4) = 0\big\}.$$

Prove that I is a maximal ideal of $\mathbb{C}[x, y, z, w]$.

4. (a) State the definition of an even element in the n-th symmetric group S_n.

(b) Let $n \geq 2$ and H be a subgroup of S_n. Prove that either all permutations in H are even or exactly half of them are even.

(c) Let pq be the order of a group G where p and q are primes with $p > q$. Assume that A is a subgroup of G and the order of A is p. Prove that A is the *unique* subgroup of order p.

Linear Algebra

A. (8 points) Let P_n be the vector space consisting of all the polynomials in the variable x over the rational numbers of degree less than n. Define a linear map $L : P_5 \to \mathbb{Q}^{2\times 3}$ by

$$L(f) = \begin{bmatrix} f(1) & 2f(-1)+f(1) & 3f(0) \\ f(1)-f(0) & f(-1)-f(1) & f(0) \end{bmatrix}.$$

(a) Specify a basis for P_5 and a basis for $\mathbb{Q}^{2\times 3}$.

(b) Find the matrix representing the linear transformation L in the bases for P_5 and $\mathbb{Q}^{2\times 3}$ you specified in (a).

(c) Using (b) or otherwise, find a basis for the kernel of L and a basis for the image of L.

B. (5 points) Let $A \in \mathbb{Q}^{10\times 10}$ be a diagonal matrix with diagonal entries

$$\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4\}.$$

Let C_A be the space of all matrices $B \in \mathbb{Q}^{10\times 10}$ which commute with A, that is,

$$C_A = \{B | AB = BA\}.$$

What is the dimension of C_A? Prove your answer.

C. (5 points) Let A be a square matrix over $\mathbb{R}$. Assume that the characteristic polynomial of A is x^7, the rank of A is 4, and the rank of A^2 is 1. Classify all such matrices A up to similarity.

D. (6 points) A real $n \times n$ matrix U is called *orthogonal* if $U^{-1} = U^T$ where U^T is the transpose of U.

(a) Prove that if U is orthogonal, then the eigenvalues of U are ± 1.

(b) Prove that the converse is true if in addition, U is symmetric.

E. (6 points) For each of the following statements, state True or False. If true, prove it. If False, give a counter example.

(a) For $A \in \mathbb{R}^{3\times 3}$, $\det(A+I) = \det(A) + \det(I)$ if $\operatorname{tr}(A) = -\operatorname{tr}\big(\operatorname{adj}(A)\big)$. (Here I denotes the 3×3 identity matrix, and $\operatorname{tr}(A)$ and $\operatorname{adj}(A)$ denote the trace and adjoint of A respectively.)

(b) A bilinear form on a vector space over the real numbers is positive definite if and only if it is non-degenerate. (A bilinear form $<,>$ is *non-degenerate* if for every $\mathbf{v} \neq 0$, there is a vector $\mathbf{w}$ with $< \mathbf{v}, \mathbf{w} > \neq 0$.)