Qualifying Examination

(*January 2007*)

- If you have any difficulty with the wording of the following problems please contact the supervisor immediately. All persons responsible for these problems, in principle, will be accessible during the entire duration of the exam.
- You are allowed to rely on a previous part of a multi-part problem even if you do not prove the previous part.
- $\mathbb{Z}$ denotes the group (or ring) of integers with the usual operations, and $\mathbb{Z}_n$ denotes the quotient group (or ring) $\mathbb{Z}/n\mathbb{Z}$.
- $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively.
- $\mathbb{C}^* = \mathbb{C} - \{0\}$ is the group with the usual multiplication.
- All matrices are over fields.

Abstract Algebra

Each part (a), (b) or (c) below is worth 3 *points, and the total points are* 30.

1. (a) How many distinct subgroups isomorphic to $\mathbb{Z}_4$ are there in $\mathbb{Z}_{24}$?

(b) Let $Z(G)$ denote the center of a group G, i.e.,

$$Z(G) = \{g \in G |\, gh = hg \text{ for all } h \in G\}.$$

Prove that if $G/Z(G)$ is cyclic then G is abelian.

(c) Prove that if p is prime then every group G of order p^2 is abelian. (Hint: you may use the fact that $Z(G)$ is not the trivial subgroup)

2. A *character* of a group G is a group homomorphism $\chi\colon G \to \mathbb{C}^*$.

(a) Prove that a character of a group G is constant on each conjugacy classs in G.

(b) Let S_n be the n-th symmetric group. For $\sigma \in S_n$, let $\operatorname{sgn}(\sigma) = (-1)^k$ if σ is the product of k transpositions. Prove that the only non-constant character of S_n is the homomorphism $\sigma \mapsto \operatorname{sgn}(\sigma)$.

3. (a) Let R be a finite domain. Prove that R is a field.

(b) Prove that if R is a principal ideal domain then every prime ideal of R is maximal.

(c) Prove or give a counter-example to the following statement:
Let R be a domain and let $a \in R$. If (a) is prime then a is irreducible.

4. (a) Let r be an element in a commutative ring R with multiplicative identity 1. Prove that if $r^{2007} = 0$ then $1 - r$ is a unit in R.

(b) Let $\alpha = \frac{1+\sqrt{-19}}{2}$. Prove that if $\phi : \mathbb{Z}[\alpha] \to \mathbb{Z}_3$ is a ring homomorphisms, then $\phi = 0$ (the zero map).

Linear Algebra

Each part (a) or (b) below is worth 3 *points, and the total points are* 30.

A. Let V be the real vector space of functions $f : \mathbb{R} \to \mathbb{R}$, and let V_e and V_o be the sets of even and odd functions $f \in V$, respectively.

(a) Show that V_e and V_o are subspaces of V;

(b) Show that $V = V_e \oplus V_o$.

B. Let V and W be vector spaces over a field F. A linear transformation $T : V \to W$ is said to be *independence preserving* if

$$T(I) = \{T(v) |\ v \in I\} \subset W$$

is linearly independent whenever $I \subset V$ is a linearly independent set.

(a) Show that T is independence preserving if T is one-to-one;

(b) Conversely, show that T is one-to-one if T is independence preserving.

C. Let A be a real matrix of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_n \end{pmatrix}$$

(a) Compute the characteristic polynomial of A;

(b) Show that the minimal polynomial and the characteristic polynomial of A are equal.

D. Let $M_n(\mathbb{R})$ be the set of all $n \times n$ real matrices, regarded as a vector space of dimension n^2 over $\mathbb{R}$.

(a) Write down explicitly a linear basis of $M_n(\mathbb{R})$;

(b) Show that for every linear transformation $T : M_n(\mathbb{R}) \to \mathbb{R}$ there exists a unique matrix $A \in M_n(\mathbb{R})$ such that $T(X) = \mathrm{Tr}(AX)$ for all $X \in M_n(\mathbb{R})$.

E. Let A be a 3×3 matrix with entries in a field F of characteristic 0.

(a) Prove that the characteristic polynomial of A is equal to

$$\lambda^3 - \mathrm{Tr}(A)\,\lambda^2 + \frac{1}{2}[\mathrm{Tr}(A)^2 - \mathrm{Tr}(A^2)]\,\lambda - \det(A);$$

(b) Assume that $\mathrm{Tr}(A) = 6$, $\mathrm{Tr}(A^2) = 14$, and $\det(A) = 6$. Prove that A is similar over F to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$