Qualifying Examination I **November 27, 1995**

If you have difficulty with the wording of any of the following problems, please contact the supervisor immediately. All persons responsible for these problems will, in principle, be accessible during the entire duration of the exam.

Algebra

1. (10 points) Determine all of the normal subgroups of S_4, the symmetric group on 4 letters. Give a detailed justification for your answer.

2. **(a)** (5 points) Let I be the ideal generated by X^3+1 and X^4+X^2+1 in the polynomial ring $\mathbb{Z}_2[X]$. Show that $\mathbb{Z}_2[X]/I$ is a field with four elements.

(b) (5 points) Let $m = 561 = 3 \cdot 11 \cdot 17$, and let a be a positive integer. If $(a, m) = 1$ (i.e., if a and m are relatively prime), show that $a^{m-1} \equiv 1 \bmod m$.

3. Let $(\mathcal{R}, +, \cdot)$ be an integral domain with n elements ($n \geq 2$).

(a) (5 points) Show that the characteristic of $\mathcal{R}$ is a prime number $p(\geq 2)$.

(b) (5 points) Show that $n = p^m$ for some integer m. (Hint: You may use the fact that, if a prime number q divides the order of a group G, then G has an element of order q.)

Linear Algebra

4. For each integer $1 \leq i \leq n$, let V_i be a finite-dimensional vector space, and for each $1 \leq i \leq n-1$ let $f_i : V_i \to V_{i+1}$ be a linear map. The sequence of linear maps

$$V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \xrightarrow{f_3} V_4 \longrightarrow \cdots \longrightarrow V_{n-1} \xrightarrow{f_{n-1}} V_n$$

is called an *exact sequence* if Range f_i = Kernel f_{i+1} for each $2 \leq i \leq n$. We consider exact sequences for which $\dim V_1 = \dim V_n = 0$,

$$0 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow \cdots \longrightarrow V_{n-1} \longrightarrow 0.$$

(a) (2 points) If $0 \longrightarrow V_2 \longrightarrow 0$ is exact, show that $\dim V_2 = 0$.

(b) (2 points) If $0 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$ is exact, show that $\dim V_2 - \dim V_3 = 0$.

(c) (2 points) If $0 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow V_4 \longrightarrow 0$ is exact, show that

$$\dim V_2 - \dim V_3 + \dim V_4 = 0.$$

(d) (4 points) If $0 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow \cdots \longrightarrow V_{n-1} \longrightarrow 0$ is exact, show that

$$\sum_{i=2}^{n-1} (-1)^i \dim V_i = 0.$$

5. Let A be a real $n \times n$ matrix.

- **(a)** (5 points) Suppose $AA = A$. Either prove that A is symmetric, or give a counterexample.
- **(b)** (5 points) Suppose $AA = A$ and that $A^T A = AA^T$. Either prove that A is symmetric, or give a counterexample.

6. Let V be a real vector space of dimension n. Suppose that $T : V \to V$ is a linear map.

- **(a)** (3 points) Let $F_T = \{\, S \in L(V, V) : ST = 0 \,\}$. Show that F_T is a subspace, and compute its dimension in terms of $\dim V$ and $\operatorname{rank} T$.
- **(b)** (3 points) Let $G_T = \{\, J \in L(V, V) : TJ = 0 \,\}$. Show that G_T is a subspace, and compute its dimension in terms of $\dim V$ and $\operatorname{rank} T$.
- **(c)** (4 points) Either prove that $F_T = G_T$ for every $T \in L(V, V)$, or give a specific counterexample.