Qualifying Examination I **April 22, 1996**

If you have difficulty with the wording of any of the following problems, please contact the supervisor immediately. All persons responsible for these problems will, in principle, be accessible during the entire duration of the exam.

Algebra

1. Determine whether or not the following statements are true or false. Back up your answer with a counterexample if the statement is false, or with a brief reason if the statement is true.

- **(a)** (2 points) If A is a ring which contains a field of characteristic $p \geq 0$ as a subring, then A/I is a field of the same characteristic p whenever I is a (proper) maximal ideal of A.
- **(b)** (2 points) A prime element in a unique factorization domain generates a maximal ideal.
- **(c)** (2 points) Suppose A is an integral domain and suppose $f(X)$ is an irreducible polynomial in $A[X]$. If p is a prime element in A, then $\overline{f}(X)$ is irreducible in $(A/pA)[X]$, where $\overline{f}(X)$ is the polynomial $f(X)$ with coefficients reduced modulo pA.
- **(d)** (2 points) If -1 is not a square in the field K, then -1 is also not a square in the field $K(X)$, where X is an indeterminate.
- **(e)** (2 points) Every element of the symmetric group S_n on n symbols has order $\leq n$, where $n = 1, 2, 3, \ldots$.

2. (10 points) Prove that the only non-abelian group of order 6 is the symmetric group S_3 of all permutations on 3 symbols.

3. Suppose that G is a finite group of order $m \cdot n$, where $(m, n) = 1$. Further suppose that G has a normal subgroup B of order n and another subgroup A of order m.

- **(a)** Prove that
  - **(i)** (3 points) $A \cdot B := \{ a \cdot b : a \in A, b \in B \}$ is a subgroup of G,
  - **(ii)** (3 points) there is a group isomorphism from A to $(A \cdot B)/B$,
  - **(iii)** (3 points) $A \cdot B = G$.
- **(b)** (1 point) Can you give an example of a group G and proper subgroups A and B of G such that $G = A \cdot B$, B is a normal subgroup of G, and $G \neq A \times B$?

Linear Algebra

4. Let M be the space of 2×2 complex matrices. For $A \in M$, consider the subspace

$$C_A = \{ B \in M : AB - BA = 0 .\}.$$

(a) (1 point) If $G \in M$ is invertible, show that

$$\dim C_{G^{-1}AG} = \dim C_A.$$

(b) (1 point) Let $I \in M$ be the identity matrix. If λ is any scalar, show that

$$\dim C_{A+\lambda I} = \dim C_A.$$

(c) (3 points) Compute $\dim C_S$ and $\dim C_T$, where

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(d) (5 points) Using the results of (a), (b), (c), and the Jordan canonical form, prove that $\dim C_A \neq 2$ if and only if A is a scalar multiple of the identity I.

5. For each $1 \leq i \leq n$, let $A_i : V \to V$ and $B_i : W \to W$ be linear maps on the complex vector spaces V and W, respectively. Let $\phi : V \to W$ be a linear map such that $\phi A_i = B_i \phi$ for each i.

(a) (1 point) Show that the kernel $\ker(\phi) \subset V$ is invariant under each A_i. (Recall that a subspace $U \subset V$ is invariant under a linear map $A : V \to V$ if $Au \in U$ whenever $u \in U$.)

(b) (1 point) Show that the image $\mathrm{im}(\phi) \subset W$ is invariant under each B_i.

(c) (4 points) Suppose that V has no proper subspace that is invariant under each A_i, and that W has no proper subspace that is invariant under each B_i. Show that $\phi : V \to W$ is either an isomorphism or the zero map.

(d) (4 points) Consider the special case $W = V$. Suppose that V has no proper subspace that is invariant under each A_i, and no proper subspace that is invariant under each B_i. Show that $\phi : V \to V$ is a scalar multiple of the identity.

6. **(a)** (5 points) Let A be a 2×2 complex matrix such that $A^2 = I$, where I is the identity matrix. Show that A is diagonalizable over the complex field $\mathbb{C}$.

(b) (5 points) Let A be an $n \times n$ complex matrix such that $A^k = I$ for some positive integer k. Show that A is diagonalizable over the complex field $\mathbb{C}$.