Qualifying Examination I **August 17, 1998**

If you have any difficulty with the wording of the following problems please contact the supervisor immediately. All persons responsible for these problems, in principle, will be accessible during the entire duration of the exam.

Algebra

S_n denotes the symmetric group on n letters.
$\mathbb{Z}$ denotes the integers.

1. **(a)** (4 points) Let R and S be rings with unity. Let $\phi : R \to S$ be a map from R to S. What properties must ϕ have to be a (unit preserving) ring homomorphism?

(b) (3 points) Let $\phi : R \to S$ be a (unit preserving) ring homomorphism. Show that $I = \ker \phi$ is a two-sided ideal of R.

(c) (3 points) What are the conditions on n and m (non-negative integers) for there to be a (unit preserving) ring homomorphism of $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$? Justify your answer.

2. **(a)** (5 points) Find explicitly (with justification) all subgroups of the group $(\mathbb{Z}/99\mathbb{Z}, +)$.

(b) (5 points) Let $\gamma \in S_n$ be an r-cycle. For $1 \leq k \leq r$, determine the order of γ^k.

3. Prove or disprove each statement.

(a) (3 points) if H is a normal subgroup of K and K is a normal subgroup of G then H is a normal subgroup of G.

(b) (3 points) If R is a commutative ring, S is a field and $\phi : R \to S$ is a (unit preserving) ring homomorphism then $\ker \phi$ is a maximal ideal of R.

(c) (4 points) Given a finite group G, there exists an integer n and injective (i.e., $1-1$) group homomorphism from G to S_n.

Linear Algebra

4. Let M be the vector space of real $n \times n$ matrices. Let $M_+ \subset M$ denote the subspace of symmetric matrices, and $M_- \subset M$ the subspace of antisymmetric matrices.

(a) (2 points) Find a basis for M, a basis for M_+, and a basis for M_-, and compute the dimension of each.

(b) (2 points) State the definition of an inner product. Then verify that $g(A, B) = Tr(A^T B)$ is an inner product on M, where A^T denotes the transpose of A and Tr denotes the trace.

(c) (2 point) Compute the dimension of the orthogonal complement $M_+^\perp$ to the subspace $M_+ \subset M$.

(d) (2 points) Show that $M_- = M_+^\perp$. (Hint: first show that $M_- \subset M_+^\perp$).

(e) (2 points) Let $D \subset M$ be the subspace consisting of diagonal matrices. Show that $M_- \subset D^\perp$.

5. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ be complex $n \times n$ matrices such that

$$\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 2\, \delta_{i,j}\, I,$$

where I is the $n \times n$ identity matrix, and $\delta_{i,j}$ equals 1 if $i = j$ and equals 0 if $i \neq j$.

(a) (2 points) Show that for any two $n \times n$ matrices A and B that the trace of AB is the same as the trace of BA.

(b) (2 points) Show that the product $\Gamma_i \Gamma_j$ has zero trace whenever $i \neq j$.

(c) (2 points) Show that the matrices $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ are linearly independent.

(d) (2 points) Now generalize the result of part (b) as follows: show that the product $\Gamma_{i_1} \Gamma_{i_2} \ldots \Gamma_{i_k}$ of an even number k of these matrices has zero trace if $i_1 < i_2 < \cdots < i_k$.

(e) (2 points) Assuming that $m > 1$, show that each matrix Γ_i has zero trace (Hint: Multiply by Γ_j^2, where $j \neq i$.)

6. Let A be the matrix

$$\begin{bmatrix} 9 & -25 & 8 \\ -15 & 15 & 0 \\ -12 & 0 & 6 \end{bmatrix}$$

(a) (2 points) Find the rank of A.

(b) (2 points) Find the characteristic polynomial of A.

(c) (2 points) Find all the eigenvalues of A, and construct a basis for the eigenspace associated to each of the eigenvalues.

(d) (2 points) Find the minimal polynomial of A.

(e) (2 points) Find the Jordan canonical form of A.