Qualifying Examination I **January 6, 1998**

If you have difficulty with the wording of any of the following problems, please contact the supervisor immediately. All persons responsible for these problems will, in principle, be accessible during the entire duration of the exam.

Algebra

1. **(a)** (4 points) Suppose that H_1 and H_2 are normal subgroups of the group G. Show that $H_1H_2 = \{ h_1h_2 : h_1 \in H_1,\ h_2 \in H_2 \}$ is a subgroup of G.

(b) (3 points) Let G be a group. For $x \in G$, let $o(x)$ denote the order of x. Show that if $o(a) = n$ and $o(b) = m$ and $\gcd(n, m) = 1$, then $< a > \cap < b >= \{e\}$. ($< x >$ is the subgroup of G generated by x, and e is the identity element of G.)

(c) (3 points) Give an example of a ring R and a prime ideal P which is not maximal.

You may use the following theorems to solve the problems below:

Theorem A. *Let p be a prime integer which divides the order of a finite group G. Then G has a subgroup of order p.*

Theorem B. *Let p be a prime integer which divides the order of a finite group G. The number of subgroups of order p is congruent to $1 \bmod p$. (e.g., if 7 divides $|G|$, then the number of subgroups of order 7 would be one of 1, 8, 15, ...)*

2. Let G be a group of order 15. We will show that G must be abelian.

(a) (3 points) Show that G has a subgroup A of order 5 and a subgroup B of order 3 and that these subgroups are normal. (Hint: If H is a subgroup of order n, then so is $g^{-1}Hg$.)

(b) (3 points) Say $A =< a >$ and $B =< b >$. Explain first why A and B are cyclic and second why it is enough to show that $ab = ba$ in order to see that G is abelian. (Hint: Problem 1 should be of some help.)

(c) (4 points) Clearly, $ab = ba$ if and only if $aba^{-1}b^{-1} = e$. Using part (a) of this problem and part (b) of problem 1, show that $aba^{-1}b^{-1} = e$.

3. (a) (3 points) Let R be a commutative integral domain. A noninvertible element $p \in R$ is called "prime" if whenever $p \mid ab$ then $p \mid a$ or $p \mid b$. A noninvertible element $u \in R$ is called irreducible if $u = xy$ implies that either x or y is invertible. Prove that if p is a prime element then p is irreducible.

(b) (3 points) Show that in the ring $\mathbb{C}[x^2, x^3] =\{$ polynomials in $\mathbb{C}[x]$ such that the coefficient of x is $0\}$, the element x^2 is irreducible, but is not prime.

(c) (4 points) Prove that if R is a finite ring with no zerodivisors then R is a division ring.

Linear Algebra

4. Let A and B be complex 2×2 matrices.

(a) (3 points) Show that A is noninvertible if and only if the characteristic polynomial is $p(t) = t^2 - k\,t$ for some constant k.

(b) (3 points) Suppose that A is noninvertible. Show that $rank(A) = 1$ if and only if the minimal polynomial equals the characteristic polynomial.

(c) (4 points) Suppose that A and B are both noninvertible, and the characteristic polynomial of A is equal to the characteristic polynomial of B. Is the Jordan canonical form of A equal to the Jordan canonical form of B? Justify your answer with a proof or with a counterexample.

5. Let

$$J_1 = \begin{pmatrix} 0 & -3 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & -3 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 3\,i & 0 & 0 \\ -i & 0 & 2\,i & 0 \\ 0 & -2\,i & 0 & i \\ 0 & 0 & -3\,i & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

(a) (3 points) Compute the characteristic polynomial, the minimal polynomial, and the Jordan canonical form of J_3.

(b) (3 points) Compute the characteristic polynomial, the minimal polynomial, and the Jordan canonical form of J_1.

(c) (4 points) Compute the characteristic polynomial, the minimal polynomial, and the Jordan canonical form of $J_1 + i\,J_2$.

6. Let M_n be the vector space of real $n \times n$ matrices. Let W be an m-dimensional subspace of $\mathbb{R}^n$, and $P \in M_n$ be the orthogonal projection matrix onto W. Compute the rank of each of the following linear maps.

(a) (1 point) $P : \mathbb{R}^n \to \mathbb{R}^n$.

(b) (1 point) $P^2 : \mathbb{R}^n \to \mathbb{R}^n$.

(c) (1 point) $P^T : \mathbb{R}^n \to \mathbb{R}^n$, where P^T is the transpose of P.

(d) (2 points) $\alpha : M_n \to \mathbb{R}$ defined by $\alpha(A) = Tr\,(A\,P)$.

(e) (2 points) $\kappa : M_n \to M_n$ defined by $\kappa(A) = P^T\,A\,P$.

(f) (3 points) $\gamma : M_n \to M_n$ defined by $\gamma(A) = P^T\,(A - A^T)\,P$.