

## Qualifying Examination

(January 2005)

### Algebra

- 1.** (10 points) A *partition* of a positive integer  $n$  is an ordered collection  $\lambda = (\lambda_1, \dots, \lambda_k)$  of integers such that  $\lambda_1 \geq \dots \geq \lambda_k > 0$  and  $\lambda_1 + \dots + \lambda_k = n$ . Let  $S_5$  denote the symmetric group of 5 letters.
- List all the partitions of 5;
  - How many conjugacy classes are there in  $S_5$ ? You must justify your answer;
  - For each conjugacy class of  $S_5$ , determine the number of elements in it. (Note that *these numbers should add up to 5!*)
- 2.** (8 points) Let  $R$  be a commutative ring. A subset  $S$  of  $R$  is *multiplicatively closed* if  $0 \notin S$  and if  $s_1 s_2 \in S$  whenever  $s_1, s_2 \in S$ .
- State the definition of prime ideal;
  - Let  $S \subset R$  be multiplicatively closed. Let  $I$  be an ideal of  $R$  which is maximal among all the ideals  $J$  of  $R$  satisfying  $J \cap S = \emptyset$ . Prove that  $I$  is a prime ideal.
  - Give an example to show that if  $0 \notin S$  but  $S \subset R$  is not multiplicatively closed, then the conclusion of (ii) may not be true.
- 3.** (12 points) (i) Up to isomorphism, how many fields have exactly 8 elements? Construct a field with 8 elements;
- Up to isomorphism, how many abelian groups have exactly 8 elements? List all of them.
  - Let  $R$  be a ring with identity. If  $R$  contains exactly 8 elements and if  $p$  denotes the characteristic of  $R$ , then show that  $p = 2, 4$  or  $8$ .

**Linear Algebra**

1. (10 points) (i) Consider the linear operator  $T : \mathbb{C}[x] \rightarrow \mathbb{C}[x]$  defined by

$$T(a_0 + a_1x + a_2x^2 + \dots + a_nx^n) = a_1 + 2a_2x + \dots + na_nx^{n-1}.$$

Determine all the eigenvalues and eigenspaces of  $T$ .

- (ii) Is the following statement True or False?

*Every linear operator  $T$  on a vector space  $V$  over  $\mathbb{C}$  has at least one eigenvalue.*

If the statement is true, prove it; if it is false, give a counter-example.

2. (10 points) (i) State the Cayley-Hamilton theorem.

(ii) Prove that the characteristic and minimal polynomials of a matrix  $A$  have the same roots (except for multiplicities).

3. (5 points) Consider the following matrix:

$$A = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ & 2 & 1 \end{pmatrix}.$$

Determine the Jordan form of  $A$ .

4. (5 points) Let  $V$  be the vector space of all  $n \times n$  matrices over  $\mathbb{C}$ , with the inner product

$$(A|B) = \text{Tr}(AB^*)$$

where  $B^*$  is the conjugate transpose of  $B$ . Find the orthogonal complement of the subspace of diagonal matrices.