Qualifying Examination in Analysis

11 August 2005

- You may ask the proctor for clarifications.
- You are allowed to rely on a previous part of a multi-part problem even if you do not prove the previous part.
- Notation: $\mathbb{R}$ denotes the real numbers and $\mathbb{C}$ the complex numbers.

Real Analysis I: One-dimensional Calculus

1. **(a)** (2 points) Let $g : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$. Show that if $c \in (a, b)$ and $g(x) \leq g(y)$ for every $x \in [a, c]$ and $y \in [c, b]$, then

$$\frac{1}{c-a}\int_a^c g(t)\,dt \leq \frac{1}{b-c}\int_c^b g(t)\,dt.$$

(b) (3 points) Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a twice continuously differentiable function so that $f''(t) \geq 0$ for every $t \in I$. Prove that f is convex on $[a, b]$; that is,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

whenever $x, y \in [a, b]$ and $\lambda \in [0, 1]$. Hint: Use part (a) with $g = f'$ and an appropriate choice of a, b, and c.

(c) (2 points) For $\{x_1, x_2, \ldots, x_n\} \subset [a, b]$ and f a convex function on $[a, b]$, prove that

$$f\Big(\frac{x_1 + x_2 + \cdots + x_n}{n}\Big) \leq \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

(d) (3 points) If $x_1, x_2, \ldots, x_n$ are positive real numbers, then

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \leq \Big(\frac{\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n}}{n}\Big)^2 \leq \frac{x_1 + x_2 + \cdots x_n}{n}.$$

2. **(a)** (2 points) State the fundamental theorem of calculus

   (b) (5 points) Suppose $f \in C^1([0,\infty))$, $f(0) = 0$, and $0 < f'(x) \le 1$ for all $x \ge 0$. Show that

$$\left(\int_0^x f(t)\,dt\right)^2 \ge \int_0^x \big(f(t)\big)^3\,dt$$

   for all $x \ge 0$. Hint: Show that the left-hand side minus the right-hand side is increasing on $[0,\infty)$.

   (c) (3 points) Show that $f(x) = x$ is the only f as in part (b) such that

$$\left(\int_0^x f(t)\,dt\right)^2 = \int_0^x \big(f(t)\big)^3\,dt$$

   for all $x \ge 0$.

Real Analysis II: Multi-dimensional Calculus

1. **(a)** (2 points) State the change of variables formula for multiple integrals.

   (b) (4 points) Let Ω be a bounded region in $\mathbb{R}^n$ with the usual coordinates $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and define, for each $i \in \{1, 2, \ldots, n\}$,

$$m_i = \frac{\int_\Omega x_i\,d\mathbf{x}}{\int_\Omega d\mathbf{x}}.$$

   The point $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ is called the centroid of Ω. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be given by $g(\mathbf{x}) = \mathbf{b} + A\mathbf{x}$, where $\mathbf{b} \in \mathbb{R}^n$ and A is a linear transformation of $\mathbb{R}^n$. (Such a function is called affine.) Prove that $g(\mathbf{m})$ is the centroid of $g(\Omega)$.

   (c) (4 points) Evaluate the integral

$$\int_{\mathbb{R}^n} e^{-x_1^2 - x_2^2 - \cdots - x_n^2}\,d\mathbf{x}.$$

   Hint: Every point in $\mathbb{R}^n$ can be represented in the form

$$\mathbf{x} = r\sum_{i=1}^n \cos\theta_i \mathbf{e}_i$$

   where r is the length of $\mathbf{x}$, $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is the usual basis of $\mathbb{R}^n$, and $\mathbf{x} \cdot \mathbf{e}_i = r\cos\theta_i \mathbf{e}_i$.

2. **(a)** (4 points) Prove that the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by $f(0, y) = 0$ and $f(x, y) = x^2 \sin(y/x) + y$ whenever $x \neq 0$ is differentiable at $(x, y) = (0, 0)$. Also, compute its derivative.

   (b) (2 points) Suppose that $F : \mathbf{R}^n \to \mathbf{R}$, F is differentiable at the origin, $\epsilon > 0$, and $DF(0) = 0$. Also, let $|x|$ denote the length of $x \in \mathbf{R}^n$. Prove that there is an open ball B centered at the origin such that

$$|F(x) - F(0)| \leq \epsilon |x|$$

   whenever $x \in B$.

   (c) (4 points) Supppose that $g : \mathbf{R}^n \to \mathbf{R}$ is differentiable at $a \in \mathbb{R}^n$ and $\epsilon > 0$. Prove that there is an open ball B centered at a such that

$$|g(x) - g(a)| \leq (|Dg(a)| + \epsilon)|x - a|$$

   whenever $x \in B$.

Complex Analysis

1. (10 points) Evaluate the (real) integral

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1 + e^x}\, dx, \quad 0 < \alpha < 1$$

   and justify each step of your procedure. Hint: Consider the rectangle in the complex plane with corners $-T$, T, $T + 2\pi i$ and $-T + 2\pi i$.

2. **For this problem f is an entire function such that $|f(z)| = 1$ whenever $|z| = 1$ and $f^{(n)}$ denotes the nth derivative of f.**

   (a) (1 point) Show that if n an m are integers, then $\int_{-\pi}^{\pi} e^{in\theta} e^{im\theta}\, d\theta$ is 2π in case $n + m = 0$ and zero otherwise.

   (b) (3 points) Show that if n is a positive integer, then

$$\frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta}\, d\theta.$$

   (c) (3 points) Prove that $\sum_{n=0}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right|^2 = 1$. Hint: Consider Taylor expansions of the function f and its conjugate.

   (d) (3 points) If $f''(0) = 2$, then $f(z) = z^2$.