Qualifying Examination in Analysis

January, 2007

- If you have any difficulty with the wording of the following problems please contact the supervisor immediately. All persons responsible for these problems, in principle, will be accessible during the entire duration of the exam.
- You are allowed to rely on a previous part of a multi-part problem even if you do not prove the previous part.
- Notation: $\mathbb{R}$ denotes the real numbers, $\mathbb{N}$ the positive integers, and $\mathbb{C}$ the complex numbers.

Real Analysis I: One-dimensional Calculus

1. **(a)** (2 points) Define what it means for a sequence to be Cauchy convergent.

   (b) (3 points) Define what it means for a series $\sum_1^\infty a_n$ to converge.

   (c) (5 points) Show that if $0 \leq a_n \leq b_n$ and $\sum_1^\infty b_n$ converges, then $\sum_1^\infty a_n$ converges.

2. Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous on $[0, \infty)$, differentiable on $(0, \infty)$, $f(0) = 0$, $0 \leq f'(x) \leq 1$ for $x > 0$.

   (a) (4 points) Show that the function

$$F(x) = 2 \int_0^x f(t)\, dt - f(x)^2$$

   is increasing on $[0, \infty)$ (i.e. $F(x_1) \leq F(x_2)$ if $0 \leq x_1 < x_2$).

   (b) (3 points) Show that for $x \geq 0$

$$2 \int_0^x f(t)\, dt \geq f(x)^2.$$

(c) (3 points) Show that for $x \geq 0$

$$\left(\int_0^x f(t)\, dt\right)^2 \geq \int_0^x f(t)^3\, dt.$$

Real Analysis II: Multi-dimensional Calculus

1. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is $C^2(\mathbb{R}^2)$, and that

$$f(0,0) = 0, \quad \frac{\partial f}{\partial x}(0,0) \neq 0, \quad \frac{\partial f}{\partial y}(0,0) = 0, \quad \frac{\partial^2 f}{\partial y^2}(0,0) \neq 0.$$

   (a) (2 points) Explain why there exists $\epsilon > 0$, and a function $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}$, such that $f(\gamma(y), y) = 0$.

   (b) (4 points) Show that $\gamma'(0) = 0$ and $\gamma''(0) \neq 0$.

   (c) (4 points) Show that the function $y \mapsto \frac{\partial f}{\partial y}(\gamma(y), y)$ changes sign at $y = 0$.

2. The second differential of a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ at $(a,b) \in \mathbb{R}^2$ is defined as

$$d^2 f((a,b),(h,k)) = h^2 \frac{\partial^2 f}{\partial x^2}(a,b) + 2hk \frac{\partial^2 f}{\partial x \partial y}(a,b) + k^2 \frac{\partial^2 f}{\partial y^2}(a,b)$$

   for any $(h,k) \in \mathbb{R}^2$.

   (a) (3 points) Let $f(x,y) = \cos x \cos y$, for $(x,y) \in \mathbb{R}^2$. Show that $|d^2 f((a,b),(h,k))| \leq (h+k)^2$, for any $(a,b) \in \mathbb{R}^2$, $h \geq 0,\ k \geq 0$.

   (b) (3 points) If $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1,\ x \geq 0,\ y \geq 0\}$, show that

$$|f(x,y) - 1| \leq \frac{1}{2}(x+y)^2, \qquad \forall (x,y) \in A$$

   (c) (1 point) Compute the area of A.

   (d) (3 points) Show that

$$\left| \int_A f(x,y)\, dx dy - \frac{\pi}{4} \right| \leq \frac{2+\pi}{16}$$

Complex Analysis

1. Suppose f and g are entire functions, and that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$.

   (a) (4 points) Show that if $g(z) \neq 0$ for all $z \in \mathbb{C}$, then there is a constant c such that $f(z) = c\, g(z)$, for all $z \in \mathbb{C}$.

   (b) (3 points) Show that if $z = 0$ is the only zero of $g(z)$, then there is a constant c such that $f(z) = c\, g(z)$, for all $z \in \mathbb{C}$.

   (c) (3 points) Show that if $g(1/n) = 0$ for $n = 1, 2, \ldots$, then there is a constant c such that $f(z) = c\, g(z)$, for all $z \in \mathbb{C}$.

2. **(a)** (2 points) State a version of the residue theorem.

   (b) (2 points) Find the residues of the function $f(z) = \dfrac{e^{iz}}{(z-1)^2 + 4}$

   (c) (6 points) Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x-1)^2 + 4}\, dx.$$