Qualifying Examination in Analysis

January, 2008

- If you have any difficulty with the wording of the following problems please contact the supervisor immediately. All persons responsible for these problems, in principle, will be accessible during the entire duration of the exam.
- You are allowed to rely on a previous part of a multi-part problem even if you do not prove the previous part.
- Notation: $\mathbb{R}$ denotes the real numbers, $\mathbb{N}$ the positive integers, and $\mathbb{C}$ the complex numbers.

Real Analysis I: One-dimensional Calculus

1. For each $n \in \mathbb{N}$ let $f_n : \mathbb{R} \to \mathbb{R}$ be defined as

$$f_n(x) = \frac{1}{1 + (x-n)^2} \cos \frac{x}{n}.$$

(a) (3 points) Show that f_n converges pointwise on $\mathbb{R}$.

(b) (4 points) Show that f_n does not converge uniformly on $\mathbb{R}$ but does converge uniformly on $(-\infty, A]$, any $A \in \mathbb{R}$.

(c) (3 points) Show that the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $(-\infty, A]$, any $A \in \mathbb{R}$.

2. Let $f : [0, \infty) \to [0, \infty)$ be differentiable and increasing, that is $f(x_1) \leq f(x_2)$ if $0 \leq x_1 < x_2$. Let $g : [0, \infty) \to (0, \infty)$ be differentiable and decreasing, that is $g(x_1) \geq g(x_2)$ if $0 \leq x_1 < x_2$. Let

$$F(x) = \int_0^x f(t)dt \qquad G(x) = \int_0^x g(t)dt, \quad x \geq 0.$$

(a) (3 points) Show that $f(x)G(x) - g(x)F(x) \geq 0$ on $[0, \infty)$. Conclude that $\dfrac{F(x)}{G(x)}$ is increasing on $[0, \infty)$.

(b) (4 points) Show that

(i) f not identically $0 \implies \lim_{x \to +\infty} F(x) = +\infty$

(ii) $\lim_{x \to +\infty} g(x) = L > 0 \implies \lim_{x \to +\infty} G(x) = +\infty$.

(c) (3 points) Show that $\lim_{x \to +\infty} \dfrac{F(x)}{G(x)}$ is finite if both $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to +\infty} g(x)$ are finite and positive.

Real Analysis II: Multi-dimensional Calculus

1. Let $f(x, y, z) = z + x^2 + xz + y^3 + z^3$, for $(x, y, z) \in \mathbb{R}^3$.

(a) (3 points) Use the Implicit Function Theorem to show the existence function $z = g(x, y)$, which is C^1 in a neighborhood of the origin, and which is implicitly defined by $f(x, y, z) = 0$. Make sure each hypothesis is stated and verified.

(b) (3 points) Verify that $\dfrac{\partial g}{\partial x}(0, 0) = \dfrac{\partial g}{\partial y}(0, 0) = 0$ and $\dfrac{\partial^2 g}{\partial x^2}(0, 0) = -2$.

(c) (4 points) Show that the origin is not a local extremum for g (Hint: look at the sign of $g(0, y)$ around the origin).

2. Let $f(x, y) = x^2 y^3 \cos \dfrac{1}{x^4 + y^4}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.

(a) (2 points) Compute the partial derivatives of f at (0,0).

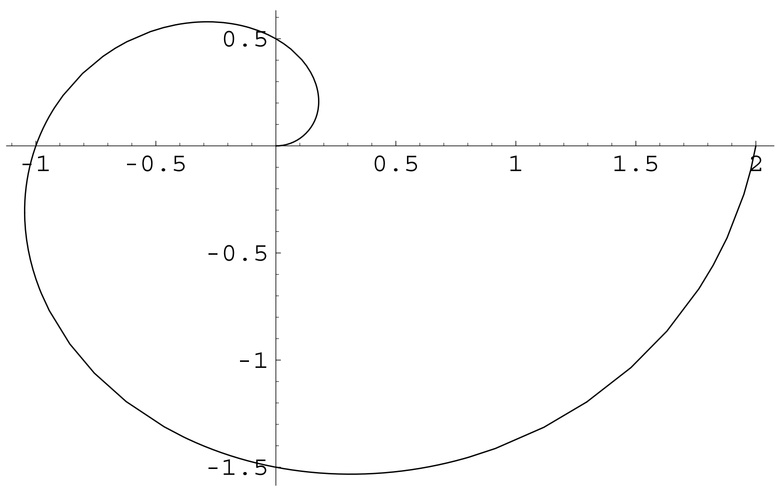
(b) (4 points) Show that f is differentiable at the origin.

(c) (4 points) Show that f is not C^1 on $\mathbb{R}^2$.

Complex Analysis

1. **(a)** (5 point) Evaluate $\int_\gamma e^{\overline{z}}\,dz$, where γ is the boundary of the square $\{(x,y) : |x| \leq 1,\ |y| \leq 1\}$, oriented counterclockwise.

   (b) (5 points) Evaluate $\int_\gamma \frac{1}{1+z^2}\,dz$, where γ is the curve with equation $z(t) = \frac{t\,e^{it}}{\pi}$, $t \in [0, 2\pi]$ (depicted in the picture below).



2. **(a)** (3 points) Suppose that f is analytic on a neighborhood of zero, except at zero. Define what it means for $z = 0$ to be (i) a removable singularity, (ii) a pole, (iii) an essential singularity for f.

   (b) (4 points) For each of the following functions classify the type of singularity at 0 (justifying your answers):

$$f(z) = \sin\frac{1}{\sin z}, \qquad g(z) = \frac{e^z - 1}{\sin^5 z}$$

   (c) (3 points) Show that if f has a simple pole at $z = 0$, then $g(z) = e^{f(z)}$ has an essential singularity at $z = 0$.