Name________________________ID # __________

QUALIFYING EXAMINATION / ANALYSIS **August 19, 2002**

• *If you have any difficulty with the wording of the following problems please contact the supervisor immediately.*
• *While dealing with a certain item of a multi-part problem, you are allowed to rely on any previous items (proved or not). Nonetheless, all individual answers should be fully justified.*

• *Throughout,* $\mathbb{R}$ *denotes the real numbers, and* $\mathbb{C}$ *denotes the complex numbers.*

Real Analysis I: One-dimensional calculus

1. (a) (3 points) Does the sequence

$$f_n(x) = \frac{x^n}{1 + x^{2n}}$$

converge uniformly on the interval $(0, 1)$? Justify your answer.

(b) (3 points) Prove that

$$\lim_{n\to\infty} \int_0^\infty \frac{x}{x+n}\, e^{-x}\, dx = 0.$$

Justify the steps in your calculation.

(c) (4 points) Consider the sequence of functions $\{F_n\}_{n=1}^\infty$ defined for $0 \leq x \leq 1$ by

$$F_n(x) = \int_{-\pi}^{\pi} \cos(x \sin\theta - n\cos\theta)\, d\theta.$$

Prove that both $\{F_n\}_{n=1}^\infty$ and $\{F_n'\}_{n=1}^\infty$ are uniformly bounded. Does the sequence $\{F_n\}_{n=1}^\infty$ contain a subsequence that converges uniformly on $[0, 1]$?

2. (a) (3 points) Assume that $f \in C^2(\mathbb{R})$ satisfies $f''(x) \geq 0$ for all $x \in \mathbb{R}$ and $f'(0) = 0$. Prove that $f(x) \geq f(0)$ for each $x \in \mathbb{R}$.

Hint: Use Taylor's formula.

(b) (4 points) Let $a_1, ..., a_n > 0$ be n given numbers. Prove that the following are equivalent:

(i) $a_1^x + a_2^x + \cdots + a_n^x \geq n$ for every $x \in \mathbb{R}$.

(ii) $a_1 a_2 \cdots a_n = 1$.

(c) (3 points) Let $a_1, ..., a_n > 0$ be n given numbers. Prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Hint: Reduce matters to the case when $a_1 a_2 ... a_n = 1$.

Real Analysis II: Multi-dimensional calculus

1. **(a)** (3 points) Let D be a compact subset of $\mathbb{R}^2$ and let $f, g : D \to \mathbb{R}$ be continuous functions. Deduce the inequality

$$\Big(\int_D f(x,y)g(x,y)\,dxdy\Big)^2 \leq \Big(\int_D f^2(x,y)\,dxdy\Big) \cdot \Big(\int_D g^2(x,y)\,dxdy\Big)$$

as a consequence of the fact that the quadratic expression

$$\mathbb{R} \ni t \mapsto \int_D [f(x,y) + t\,g(x,y)]^2\,dxdy \in \mathbb{R}$$

does not change sign.

(b) (4 points) Let D denote the closed unit disk centered at $(0,0)$ in $\mathbb{R}^2$. Prove that if $f : D \to \mathbb{R}$ is a continuous function, then

$$\Big|\int_D e^{x^2+y^2} f(x,y)\,dxdy\Big| \leq \sqrt{\frac{\pi}{2}(e^2-1)}\,\Big(\int_D f^2(x,y)\,dxdy\Big)^{1/2}.$$

(c) (3 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with the property that $\int_0^1 |f'(x)|^2 dx \leq 4$. Prove that

$$|f(x) - f(y)| \leq 2\sqrt{|x-y|}, \qquad \text{for every } \; x, y \in (0,1).$$

Hint: Use the Fundamental Theorem of Calculus.

2. **(a)** (2 points) Give a precise statement for Taylor's theorem regarding the expansion of a function $f : \mathbb{R}^n \to \mathbb{R}$ in a power series up to order two.

(b) (3 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ have continuous partial derivatives of order ≤ 2 and consider the function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(x) = \int_{-1}^{1} f(x,t)\,dt.$$

Prove that F is differentiable and

$$F'(x) = \int_{-1}^{1} (\partial_x f)(x,t)\,dt.$$

Hint: Use the definition of the derivative.

(c) (5 points) Suppose that a and x are real variables. Show that there is some $\epsilon > 0$ such that, for each a with $|a| < \epsilon$, the equation

$$\int_0^1 e^{t^3 x} - e^{-at^2(x-1)}\,dt = 0$$

has a solution x with $|x| < 1$.

Hint: Apply the Implicit Function Theorem.

Complex Analysis

1. For the duration of this problem, assume that $f : \mathbb{C} \to \mathbb{C}$ is an entire function, and denote by D the (open) disk of radius $R > 0$, centered at the origin in $\mathbb{C}$.

(a) (3 points) Starting with Cauchy's formula in D find an integral representation formula for $f^{(n)}(z)$, the n-th derivative of f, $n \geq 0$, for $z \in D$.

(b) (4 points) Prove the estimate

$$|f^{(n)}(0)| \leq C_n \, R^{-n} \sup\{|f(\zeta)| : \zeta \in \mathbb{C}, \ |\zeta| = R\},$$

where the constant C_n depends exclusively on n. Find an explicit formula for C_n.

(c) (3 points) Suppose that there exists a non-negative integer N and a constant $C > 0$ such that

$$|f(z)| \leq C|z|^N, \qquad \text{for every } z \text{ with } |z| \geq 1.$$

Show that $f^{(n)}(0) = 0$ if $n > N$ and conclude that $f(z)$ is a polynomial in z of degree at most N.

2. Decide whether the following statements are true or false. If the statement is true, justify it (eventually quote a theorem that implies it); if it is false, give a counterexample.

(a) (2 points) If f is a holomorphic function defined on the open unit disk D centered at the origin in $\mathbb{C}$, such that $|f(z)| \leq 1$ for every $z \in D$, $f(0) = 0$ and $|f'(0)| = 1$, then there is an entire function F such that the restriction of F to D is f.

(b) (2 points) If $f : \mathbb{C} \to \mathbb{C}$ is entire and has the property that there exists $M > 0$ so that $|f(z)| \geq M$ for each $z \in \mathbb{C}$, then f is a constant.

(c) (2 points) Every harmonic function of a complex variable is holomorphic.

(d) (2 points) Let D denote the unit disk centered at the origin in $\mathbb{C}$, and let $\bar{D}$ denote its closure in the complex plane. Suppose that f is holomorphic in some open set containing $\bar{D}$ and $f(\bar{D}) \subset D$. Then the equations $f(z) - z = 0$ and $f(z) + z = 0$ have the same number of solutions in D.

(e) (2 points) If f is a nonconstant holomorphic function in an open subset U of the complex plane (not necessarily connected) then $|f|$ has no local maximum in U.