Name______________________________ID # __________

QUALIFYING EXAMINATION / ANALYSIS **August 11, 2003**

• *If you have any difficulty with the wording of the following problems please contact the supervisor immediately.*
• *While dealing with a certain item of a multi-part problem, you are allowed to rely on any previous items (proved or not). Nonetheless, all individual answers should be fully justified.*

• *Throughout,* $\mathbb{R}$ *denotes the real numbers, and* $\mathbb{C}$ *denotes the complex numbers.*

Real Analysis I: One-dimensional calculus

1. **(a)** (`3 points`) Prove that a sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ converges (to a finite limit) if and only the sequences

$$a_n := \min\{x_n, x_{n+1}\}, \quad b_n := \max\{x_n, x_{n+1}\}, \qquad n = 1, 2, ...,$$

converge to a common, finite limit.

(b) (`4 points`) Suppose $f : (0, 1] \to \mathbb{R}$ is a given function and $\{x_n\}_{n=1}^{\infty}$ is a given sequence of real numbers, such that $\lim_{n\to\infty} x_n = L \in \mathbb{R}$, and

$$\min\{x_n, x_{n+1}\} \le f(x) \le \max\{x_n, x_{n+1}\}, \quad \text{whenever } x \in [\tfrac{1}{n+1}, \tfrac{1}{n}],$$

for $n = 1, 2, ...$ Prove that

$$\lim_{x\to 0^+} f(x) = L.$$

(c) (`3 points`) Given a sequence of real numbers, $\{x_n\}_{n=1}^{\infty}$, prove that there exists a continuous function $f : [0, 1] \to \mathbb{R}$ such that $f(1/n) = x_n$, $n = 1, 2, ...,$ if and only if $\lim_{n\to\infty} x_n$ exists and is finite.

2. Let $f : [0, 1] \to (0, \infty)$ be a continuous function such that

$$L := \lim_{n\to\infty} \int_0^1 (f(x))^n \, dx$$

exists and is finite.

(a) (`5 points`) Prove that $L \in [0, 1]$.

(b) (`3 points`) Given an arbitrary number $a \in [0, 1]$, construct a continuous function $f : [0, 1] \to (0, \infty)$ such that $L = a$ (where L is defined as above).

(c) (`2 points`) Does the conclusion in point (a) (i.e., that $L \in [0, 1]$) remain true if the continuous function f takes values in $\mathbb{R}$, instead of $(0, \infty)$? Justify your answer.

Hint for (a): Find a good upper bound for f, first.

Real Analysis II: Multi-dimensional calculus

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ and assume that $\{(x_n, y_n)\}_{n=1}^{\infty}$ is a sequence of points in $(\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$ which converges to $(0,0)$.

(a) (`3 points`) Prove that there exists a sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$ of points in $\mathbb{R}^2 \setminus \{(0,0)\}$ which converges to $(0,0)$ and such that

$$f(a_n, b_n) = \tfrac{1}{2}[f(x_n, 0) + f(0, y_n)], \qquad n = 1, 2, ...$$

(b) (`5 points`) Suppose that, in addition, there exist two numbers $a, b \in \mathbb{R}$ such that

$$\lim_{n\to\infty} f(x_n, 0) = a, \qquad \lim_{n\to\infty} f(0, y_n) = b, \text{ and}$$

$$\lim_{(x,y)\to(0,0)} (f(x,y) - a) \cdot (f(x,y) - b) = 0.$$

Prove that $a = b$.

(c) (`2 points`) Under the above assumptions, prove that f has a limit at $(0,0)$.

Hint for (a): Use the Intermediate Value Theorem.

2. **(a)** (`2 points`) Give a precise statement for the Mean Value Theorem for a function $f : \mathbb{R}^2 \to \mathbb{R}^2$.

(b) (`4 points`) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x,y) = (\sin x \cos y, (1 + x^2 + y^2)^{-1})$. Prove that f is Lipschitz in $\mathbb{R}^2$. That is, show that there is a finite constant $L > 0$ such that

$$\|f(x_1, y_1) - f(x_2, y_2)\| \le L\|(x_1, y_1) - (x_2, y_2)\|$$

for every $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Here $\|\cdot\|$ is the standard Euclidean norm in $\mathbb{R}^2$.

(c) (`4 points`) Show that there exists an open neighborhood U of $f(0, \pi)$ in $\mathbb{R}^2$ such that f has a continuously differentiable inverse f^{-1} on U, and that f^{-1} is Lipschitz in U.

Complex Analysis

1. **(a)** (`2 points`) State Rouché's theorem and Schwarz's lemma for holomorphic functions.

(b) (`4 points`) Let $\Omega \subset \mathbb{C}$ be an open set such that $\bar{D} := \{z \in \mathbb{C} : |z| \leq 1\} \subset \Omega$. Suppose that f is holomorphic in $D := \{z \in \mathbb{C} : |z| < 1\}$, continuous on $\bar{D}$ and such that $|f(z)| < 1$ whenever $|z| = 1$.

Prove that f has a unique fixed point in D (recall that z is called a fixed point for f if $f(z) = z$).

(c) (`4 points`) Suppose that f is holomorphic in D and $|f(z)| < 1$ for all $z \in D$. Prove that if f has two distinct fixed points in D then $f(z) = z$ for every $z \in D$.

2. **(a)** (`2 points`) Give a precise statement of the Residue theorem.

(b) (`8 points`) Use the residue calculus to compute the integral

$$\int_0^{2\pi} \frac{1}{\gamma + \cos\theta}\, d\theta$$

in the case when $\gamma \in \mathbb{R}$, $\gamma > 1$.

Hint for (b): Make the substitution $z = e^{i\theta}$.