Name____________________ID # __________

QUALIFYING EXAMINATION / ANALYSIS **January 13, 2003**

• *If you have any difficulty with the wording of the following problems please contact the supervisor immediately.*
• *While dealing with a certain item of a multi-part problem, you are allowed to rely on any previous items (proved or not). Nonetheless, all individual answers should be fully justified.*

• *Throughout,* $\mathbb{R}$ *denotes the real numbers, and* $\mathbb{C}$ *denotes the complex numbers.*

Real Analysis I: One-dimensional calculus

1. (a) (2 points) State the Stone-Weierstrass approximation theorem.

(b) (4 points) Use integration by parts to prove that

$$\lim_{n\to\infty}\int_0^1 nx^{n-1}f(x)\,dx = f(1)$$

for any $f:\mathbb{R}\to\mathbb{R}$ which is continuously differentiable.

(c) (4 points) Show that the same is true if $f:\mathbb{R}\to\mathbb{R}$ is merely continuous.

Hint: Either use (a)-(b), or reason directly by splitting $\int_0^1 = \int_0^a + \int_a^1$ for some appropriate $a \in (0,1)$.

2. (a) (3 points) Give the definition of Riemann integrability for a function $f:[0,1]\to\mathbb{R}$.

(b) (2 points) Show that $f:[0,1]\to\mathbb{R}$,

$$f(x) = \begin{cases} \sin(1/x), & x>0, \\ 0, & x=0, \end{cases}$$

is not continuous.

(c) (5 points) Any continuous function on a closed, finite interval is Riemann integrable. Show that, in spite of being discontinuous, the function f above is Riemann integrable on $[0,1]$.

Hint: Split $[0,1]$ into $[0,\delta]$ and $[\delta,1]$ for some appropriately small $\delta>0$.

Real Analysis II: Multi-dimensional calculus

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function which is differentiable on $\mathbb{R}^n \setminus \{0\}$.

(a) (3 points) Prove that for each $x \in \mathbb{R}^n \setminus \{0\}$ there exits $y \in \mathbb{R}^n \setminus \{0\}$ such that

$$f(x) - f(0) = Df(y) \cdot x$$

where $Df(y)$ is the derivative of f at y, and 'dot' stands for the scalar/dot product in $\mathbb{R}^n$.

(b) (5 points) Show that if the limit

$$L = \lim_{x \to 0} Df(x)$$

exists in $\mathbb{R}^n$ then f is differentiable at $0 \in \mathbb{R}^n$ and $Df(0) = L$.

(c) (2 points) Show that a similar conclusion as in **(b)** above holds for vector-valued functions as well. More specifically, prove that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, differentiable on $\mathbb{R}^n \setminus \{0\}$ and $L = \lim_{x \to 0} Df(x)$ exists in $\mathbb{R}^{mn}$, then f is differentiable at $0 \in \mathbb{R}^n$ and $Df(0) = L$.

2. **(a)** (2 points) Give a precise statement for the Inverse Function Theorem.

(b) (5 points) Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^n$ be a continuously differentiable function. Prove that if $\mathcal{J}_f(x)$, the Jacobian determinant of f at x, is nonzero at every $x \in U$, then $f(U)$ is open.

(c) (3 points) Give an example of an open set $U \subset \mathbb{R}^2$ and $f : U \to \mathbb{R}^2$ which is continuously differentiable, with $\mathcal{J}_f(x) \neq 0$ for every $x \in U$, and yet f is not one-to-one.

Complex Analysis

1. Assume that f is a holomorphic function defined in a neighborhood of the origin and $f'(0) \neq 0$.

(a) (4 points) Show that there exists $r > 0$ such that the unique solution z of the equation $f(z) = f(0)$ in the disk $|z| < r$ is $z = 0$.

(b) (6 points) Prove that if $r > 0$ is sufficiently small, then

$$\frac{1}{f'(0)} = \frac{1}{2\pi i} \int_{|z|=r} \frac{1}{f(z) - f(0)}\, dz.$$

2. **(a)** (2 points) Give a precise statement of the Maximum Modulus Principle for holomorphic functions.

(b) (5 points) Let $D = \{z : |z| < 1\}$ be the open unit disk in the complex plane and consider a holomorphic function $f : D \to D$ such that

$$f(0) = f'(0) = ... = f^{(n)}(0) = 0$$

for some nonnegative integer n.

Prove that if f extends continuously to the closure of D, i.e. $\{z : |z| \leq 1\}$, then

$$|f(z)| \leq |z|^{n+1} \qquad \text{for every } z \in D.$$

(c) (3 points) Use a limiting argument to show that the estimate above remains valid without the extra assumption that f extends continuously to the closure of D.