Name________________________________ID # __________

QUALIFYING EXAMINATION / ANALYSIS **August 9, 2004**

- *If you have any difficulty with the wording of the following problems please contact the supervisor immediately.*
- *While dealing with a certain item of a multi-part problem, you are allowed to rely on any previous items (proved or not). Nonetheless, all individual answers should be fully justified.*

- *Throughout,* $\mathbb{R}$ *denotes the real numbers, and* $\mathbb{C}$ *denotes the complex numbers.*

Real Analysis I: One-dimensional calculus

1. Consider a continuous function $f : [0,1) \to \mathbb{R}$ with $f(0) = 0$.

(a) (`3 points`) Suppose that there exists a point $x_0 \in (0,1)$ such that $f(x_0) < 0$ and set

$$E = \{x \in [0, x_0) \,:\, f < 0 \text{ on } (x, x_0)\}, \qquad x^* = \inf E.$$

Prove that $x^* \in [0,1)$ is well defined and $f(x^*) = 0$.

(b) (`4 points`) Assume that the right-hand derivative of f,

$$D^+f(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

exists and satisfies $D^+f(x) > 0$ at every point $x \in [0,1)$. Retaining the assumptions made in the preamble of this problem, prove that $f(x) \geq 0$ for every $x \in [0,1)$.

Hint: Reason by contradiction and use (a).

(c) (`3 points`) Show that the same conclusion as in (b) holds if we only assume $D^+f(x) \geq 0$ at every point $x \in [0,1)$.

Hint: Consider the function $f(x) + \varepsilon x$, $\varepsilon > 0$, instead of $f(x)$.

2. **(a)** (`2 points`) Given a continuous function $f : (0,\infty) \to \mathbb{R}$, explain the meaning of the improper integral $\displaystyle\int_0^\infty f(x)\,dx$.

(b) (`4 points`) Prove that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent.

(c) (`4 points`) Prove that $\displaystyle\int_0^\infty \left|\frac{\sin x}{x}\right|\,dx$ is divergent.

Hint for (b) and (c): Analyze the behavior of $(\sin x)/x$ on each half of an interval of the form $[n\pi, (n+2)\pi]$, $n = 0, 2, 4, \ldots$.

Real Analysis II: Multi-dimensional calculus

1. Throughout this problem, $\mathrm{vol}(E)$ stands for the Euclidean volume of the set $E \subset \mathbb{R}^n$.

(a) (`3 points`) Assuming that $F : [-\frac{1}{2}, \frac{1}{2}]^n \to \mathbb{R}$ is continuous and $\delta > 0$ is arbitrary, prove that

$$\mathrm{vol}\Big(\{x \in [-\tfrac{1}{2}, \tfrac{1}{2}]^n \, : \, |F(x)| > \delta\}\Big) \le \frac{1}{\delta^2} \int_{[-\frac{1}{2},\frac{1}{2}]^n} |F(x)|^2 \, dx.$$

(b) (`3 points`) For each $\delta > 0$ define

$$A_n(\delta) = \{x = (x_1, ..., x_n) \in [-\tfrac{1}{2}, \tfrac{1}{2}]^n \, : \, |\tfrac{x_1 + \cdots + x_n}{n}| > \delta\}$$

Prove that, for each $\delta > 0$ fixed, $\lim_{n\to\infty} \mathrm{vol}(A_n(\delta)) = 0$.

Hint: Try to estimate the volume of $A_n(\delta)$ by making use of (a).

(c) (`2 points`) If

$$B_n(\delta) = \{x = (x_1, ..., x_n) \in [0, 1]^n \, : \, |\tfrac{x_1 + \cdots + x_n}{n} - \tfrac{1}{2}| > \delta\}$$

show that $\mathrm{vol}(A_n(\delta)) = \mathrm{vol}(B_n(\delta))$ for each $\delta > 0$.

(d) (`2 points`) Let $f : [0, 1] \to \mathbb{R}$ be continuous. Prove that

$$\lim_{n\to\infty} \int_{[0,1]^n} f\left(\frac{x_1 + \cdots + x_n}{n}\right) dx_1 ... dx_n = f(\tfrac{1}{2}).$$

Hint: For a fixed $\delta > 0$ to be determined later, split the domain of integration into $B_n(\delta)$ and its complement. For the first term, use the fact that $\mathrm{vol}(B_n(\delta)) \to 0$. For the second term, use the continuity of $f(x)$ at the point $\frac{1}{2}$.

2. **(a)** (`3 points`) State the general form of the Implicit Function Theorem.

(b) (`5 points`) Let L be an $n \times n$ matrix with real entries such that $\det L \neq 0$. Also, set $\mathbf{1} = (1, 1, ..., 1) \in \mathbb{R}^n$ and $\mathbf{0} = (0, 0, ..., 0) \in \mathbb{R}^n$. Show that there exists $\varepsilon > 0$ such that the equation

$$L\mathbf{x} + \cos(\|\mathbf{x}\|^{\mathbf{2}})\, \mathbf{a} - \cos(\mathbf{a} \cdot \mathbf{x})\, \mathbf{1} = \mathbf{0}$$

has a solution $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$ for each given $\mathbf{a} = (a_1, ..., a_n)$ with $|a_j - 1| < \varepsilon$, $j = 1, 2, ..., n$. Here, $\| \; \|$ and 'dot' stand, respectively, for the usual Euclidean norm and scalar product of vectors in $\mathbb{R}^n$.

Hint: Consider the points $\mathbf{x} = \mathbf{0}$ and $\mathbf{a} = \mathbf{1}$.

(c) (`2 points`) Justify the existence of a $\mathbb{R}^n$-valued, C^1 function g defined in a neighborhood of the point $\mathbf{1}$ such that $g(\mathbf{1}) = \mathbf{0}$ and which satisfies

$$Lg(\mathbf{a}) + \cos(\|\mathbf{g}(\mathbf{a})\|^{\mathbf{2}})\, \mathbf{a} - \cos(\mathbf{a} \cdot \mathbf{g}(\mathbf{a}))\, \mathbf{1} = \mathbf{0}$$

for each $\mathbf{a}$ near $\mathbf{1}$. Also, compute its derivative, $Dg(\mathbf{1})$.

Complex Analysis

1. **(a)** (`4 points`) Let $z_0 \in \mathbb{C}$, $R > 0$, and assume that $f : \{z \in \mathbb{C} : |z - z_0| < R\} \to \mathbb{C}$ is holomorphic. State Cauchy's Integral Representation Theorem for f and use it to prove that

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \int_{|\xi - z_0| = r} \frac{f(\xi)}{(\xi - z)^{n+1}}\, d\xi, \qquad n = 0, 1, ...,$$

for each $r \in (0, R)$ and z with $|z - z_0| < r$. Here, $f^{(n)}(z)$ stands for the n-th (complex) derivative of f at z.

(b) (`3 points`) Prove that for each $r \in (0, R)$ and $n = 0, 1, ...,$

$$\left| \frac{f^{(n)}(z_0)}{n!} \right| \leq \frac{1}{r^n} \max\left\{ |f(\xi)| : |\xi - z_0| = r \right\}.$$

(c) (`3 points`) Use the estimate above to show that the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

converges for each $z \in \mathbb{C}$ with $|z - z_0| < R$.

2. **(a)** (`4 points`) Assume that $z_0 \in \mathbb{C}$, $R > 0$, and $f : \{z \in \mathbb{C} : |z - z_0| < R\} \to \mathbb{C}$ is a nonconstant holomorphic function with $f(z_0) = 0$. Prove that there exist a positive integer k and a holomorphic function $g : \{z \in \mathbb{C} : |z - z_0| < R\} \to \mathbb{C}$ with $g(z_0) \neq 0$, such that

$$f(z) = (z - z_0)^k g(z) \qquad \text{for all } z \text{ with } |z - z_0| < R.$$

Call k the multiplicity of the zero z_0 of f.

Hint: Use the Taylor series expansion of f at z_0.

(b) (`6 points`) Suppose now that f is a holomorphic function in $\mathbb{C}$ and that Γ is a simple, closed curve in the complex plane such that $f(z) \neq 0$ at each $z \in \Gamma$. Prove that

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)}\, dz$$

is equal to the number of zeroes of f, counted with multiplicities, that are surrounded by Γ.

Hint: Use the factorization derived in (b) near each zero of f.