Name________________________________ID # __________

QUALIFYING EXAMINATION / ANALYSIS **January 12, 2004**

• *If you have any difficulty with the wording of the following problems please contact the supervisor immediately.*
• *While dealing with a certain item of a multi-part problem, you are allowed to rely on any previous items (proved or not). Nonetheless, all individual answers should be fully justified.*

• *Throughout, $\mathbb{R}$ denotes the real numbers, and $\mathbb{C}$ denotes the complex numbers.*

Real Analysis I: One-dimensional calculus

1. Suppose that $h : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $h(0) = 1$. Also, assume that $g : \mathbb{R} \to \mathbb{R}$ is continuous, satisfies $g(0) = 0$, and is differentiable on $(-\infty, 0) \cup (0, +\infty)$. Furthermore, suppose that there exist constants $C > 0$ and $\varepsilon > 0$ such that

$$|g'(x)| \leq \frac{C}{|x|^{1-\varepsilon}}, \qquad \text{for each } x \neq 0.$$

Finally, define

$$f(x) := x\, h(g(x)), \qquad x \in \mathbb{R}.$$

(a) (`4 points`) Prove that f is differentiable at $x = 0$ and determine $f'(0)$.

(b) (`6 points`) Prove that $f'(x)$ exists at every point $x \in \mathbb{R}$ and that $f' : \mathbb{R} \to \mathbb{R}$ is continuous at $x = 0$.

2. **(a)** (`3 points`) Assume that $c \in \mathbb{R}$ and that the function $f : (c, +\infty) \to \mathbb{R}$ is continuously differentiable. If $f(x) \to -\infty$ as $x \to c^+$, prove that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \to c^+$ and $f'(x_n) \to +\infty$ as $n \to \infty$.

(b) (`3 points`) If $g : [a, b] \to \mathbb{R}$ is continuous, not identically zero, and $g(a) = 0$, prove that there exist two numbers $a \leq c < d \leq b$ such that $g(c) = 0$ and $g(x) \neq 0$ for each $x \in (c, d)$.

(c) (`4 points`) Suppose $g : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b), and that there exists a constant $M > 0$ such that

$$|g'(x)| \leq M|g(x)|, \qquad \text{for all } x \in (a, b).$$

If $g(a) = 0$, prove that $g(x) = 0$ for every $x \in [a, b]$.

Hint for (c): Reason by contradiction and consider $f(x) := \ln |g(x)|$ on a suitable interval.

Real Analysis II: Multi-dimensional calculus

1. **(a)** (`3 points`) Define what it means for a function $f : [0,1]^n \to \mathbb{R}$ to be Riemann integrable.

(b) (`7 points`) Suppose that $f, g : [0,1] \to [0,+\infty)$ are Riemann integrable. Prove that

$$F(x,y) := f(x)g(y), \qquad 0 \le x, y \le 1,$$

is Riemann integrable on $[0,1] \times [0,1]$.

2. **(a)** (`3 points`) Prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and has bounded partial derivatives then it is uniformly continuous.

(b) (`5 points`) Let $f : \mathbb{R}^n \to \mathbb{R}$, $f(0) = 0$, be continuous at 0 and subadditive, i.e.

$$f(x_1 + x_2) \le f(x_1) + f(x_2), \qquad \text{for any } x_1, x_2 \in \mathbb{R}^n.$$

Prove that f is uniformly continuous.

(c) (`2 points`) Give an example of a function $f : \mathbb{R}^n \to \mathbb{R}$, n arbitrary, that is not linear and satisfies the hypotheses of (b).

Complex Analysis

1. **(a)** (`4 points`) Does there exist a non-constant holomorphic function $f = u + i\,v : \mathbb{C} \to \mathbb{C}$, with u, v real-valued, such that $u(z) = g(v(z))$, $z \in \mathbb{C}$, for some differentiable function $g : \mathbb{R} \to \mathbb{R}$?

(b) (`3 points`) Is there a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that $f(\frac{1}{n^2}) = \frac{1}{n}$ for $n = 1, 2, 3, ...$?

(c) (`3 points`) Show that if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and even, i.e. $f(-z) = f(z)$ for every $z \in \mathbb{C}$, then there exists $g : \mathbb{C} \to \mathbb{C}$ holomorphic such that $g(z^2) = f(z)$ for every $z \in \mathbb{C}$.

Hint for (c): Consider the Taylor series expansion of f.

2. For each integer $n \geq 2$, consider the equation $z^n + z + n = 0$, $z \in \mathbb{C}$.

(a) (`4 points`) Show that if k is an integer with $1 \leq k < n$, then inside the unbounded sector

$$S_k = \left\{ z \in \mathbb{C} : \ 0 < \operatorname{Arg} z < \frac{2\pi k}{n} \right\}$$

there are exactly k roots of the above equation. As usual, $\operatorname{Arg} z$ denotes the principal argument of z.

(b) (`3 points`) Show that for $n \geq 2$ all roots of the equation are inside the annulus

$$\left\{ z \in \mathbb{C} : \ \sqrt[n]{n-2} < |z| < \sqrt[n]{n+3} \right\}.$$

(c) (`3 points`) Conclude from parts (a) and (b) that there exists a sequence of complex numbers $\{z_n\}_{n=2}^{\infty}$ such that z_n solves the above equation for each $n \geq 2$ and $\lim_{n \to \infty} z_n = 1$.

Hint for (a): Prove (and use) the inequality $x^n + n > x$ that holds for all real numbers $x \geq 0$.