Name________________________________ID # __________

QUALIFYING EXAMINATION / ANALYSIS **January 10, 2005**

• *If you have any difficulty with the wording of the following problems please contact the supervisor immediately.*
• *While dealing with a certain item of a multi-part problem, you are allowed to rely on any previous items (proved or not). Nonetheless, all individual answers should be fully justified.*

• *Throughout, $\mathbb{R}$ denotes the real numbers, and $\mathbb{C}$ denotes the complex numbers.*

Real Analysis I: One-dimensional calculus

1. **(a)** (`1 point`) State the Mean Value Theorem for a real-valued function defined on a closed interval on the real line.

(b) (`3 points`) Assume that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b), and that $M = \sup \{|f'(x)| : x \in (a, b)\}$ is finite. Prove that

$$\int_a^b |f(a) - f(x)|\, dx \le \frac{M(b-a)^2}{2}.$$

(c) (`6 points`) Assume that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$ and set $M = \sup \{|f'(x)| : x \in [0, 1]\}$. Prove that, for each positive integer n,

$$\left| \sum_{j=0}^{n-1} \frac{f(j/n)}{n} - \int_0^1 f(x)\, dx \right| \le \frac{M}{2n}.$$

2. **(a)** (`1 point`) State the definition of uniform continuity.

(b) (`5 points`) Let $f : [0, \infty) \to \mathbb{R}$ be continuous and introduce

$$\omega(\delta) = \sup \{|f(x) - f(y)| : x, y \ge 0,\ \ |x - y| < \delta\}, \qquad \delta > 0,$$

$$F(x) = \int_0^x f(t)\, dt, \qquad x > 0.$$

Prove that for every $x > 0$ and $\delta > 0$,

$$|f(x)| \le \omega(\delta) + \left| \frac{F(x+\delta) - F(x)}{\delta} \right|.$$

(c) (`4 points`) Show that if $f : [0, \infty) \to \mathbb{R}$ is uniformly continuous and $\lim_{T\to\infty} \int_0^T f(t)\, dt$ exists and is finite then $\lim_{x\to\infty} f(x) = 0$.

Real Analysis II: Multi-dimensional calculus

1. **(a)** (`3 points`) State the Change of Variable Theorem for multidimensional integrals.

(b) (`2 points`) For each $t \in \mathbb{R}$ define the function

$$g^t(u, v) = (u \cos t - v \sin t\,,\ u \sin t + v \cos t), \qquad (u, v) \in \mathbb{R}^2.$$

For each fixed $t \in \mathbb{R}$ find the inverse function of $g^t : \mathbb{R}^2 \to \mathbb{R}^2$.

(c) (`5 points`) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable and let Ω stand for a "nice" region in $\mathbb{R}^2$. For each $t \in \mathbb{R}$, let Ω_t be the image of Ω under the mapping g^t (i.e., $\Omega_t = g^t(\Omega)$) and, finally, introduce

$$F(t) = \iint_{\Omega_t} f(x, y)\, dxdy.$$

Prove that for $t \in \mathbb{R}$

$$F'(t) = \iint_{\Omega_t} \left(x \frac{\partial f}{\partial y}(x, y) - y \frac{\partial f}{\partial x}(x, y) \right) dxdy.$$

Remark. You are allowed to differentiate under the integral sign (without proof).

2. **(a)** (`2 points`) State the Implicit Function Theorem.

(b) (`5 points`) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable functions and assume that $x_0 \in \mathbb{R}$ is such that $f(x_0) = 0$ and $f'(x_0) \neq 0$. Prove that the equation $f(x) = t\, g(x)$ has a unique solution $x = x(t)$, defined on an open interval of the real line containing the origin, which satisfies $x(0) = x_0$.

(c) (`3 points`) If the function $x(t)$ defined above has a local extremum at the origin then it is a constant function.

Complex Analysis

1. Let k be a fixed, positive integer and denote by C the boundary of the disk of radius $k+1$ centered at the origin of the complex plane.

(a) (`2 points`) Prove that

$$\frac{1}{z(z-1)(z-2)\cdots(z-k)} = \sum_{j=0}^{k} \frac{(-1)^{k-j}}{j!(k-j)!} \cdot \frac{1}{z-j}.$$

(b) (`3 points`) Compute

$$\int_C \frac{1}{z(z-1)(z-2)\cdots(z-k)}\, dz.$$

(c) (`5 points`) Show that

$$\int_C \frac{(z-1)(z-2)\cdots(z-k)}{z^2}\, dz = 2\pi\, i\, (-1)^{k-1} k! \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right).$$

2. **(a)** (`2 points`) Give an example of a power series $\sum_{n=0}^{\infty} a_n z^n$, with $a_0 = 1$, $a_n \in \mathbb{R}$, which converges for every $z \in \mathbb{C}$ and which sums to zero for infinitely many values of z.

(b) (`4 points`) Let $\sum_{n=0}^{\infty} a_n z^n$ be a series with complex coefficients whose radius of convergence is $R > 0$. Prove that $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ is an entire function.

(c) (`4 points`) Show that for any $0 < r < R$ there exists $M > 0$ such that

$$|f(z)| \le M\, e^{|z|/r} \quad \text{for each} \quad z \in \mathbb{C}.$$