Qualifying Examination II December 4,1995

If you have difficulty with the wording of any of the following problems, please contact the supervisor immediately. All persons responsible for these problems will, in principle, be accessible during the entire duration of the exam.

Advanced Calculus I

1. Let $f : [-1, 1] \to \mathbb{R}$ be a function such that f and its first derivative f' are continuous on $[-1, 1]$ and such that f'' exists on $(-1, 1)$. Assume that $|f''(x)| \le 1$ for all x in $(-1, 1)$.

(a) (4 points) Make use of Taylor's Theorem to show that

$$|f(x) + f(-x) - 2f(0)| \le x^2$$

for all x in $[-1, 1]$.

(b) (3 points) Prove that

$$\left| \int_{-1}^{1} f(x)\,dx - 2f(0) \right| \le \frac{1}{3}.$$

(c) (3 points) If in addition you assume that $|f(x)| \le 1$ for all x in $[-1, 1]$, show that

$$|f'(x)| \le 2$$

for all x in $[-1, 1]$.

2. (a) (3 points) For each $n \ge 1$, let $f_n(x) = \frac{n^2x^2}{e^{nx}}$. Show that the sequence $\{f_n\}_{n\ge1}$ converges pointwise to the zero function on $[0, \infty)$.

(b) (3 points) Show that the sequence $\{f_n\}_{n\ge1}$ does not converge uniformly to the zero function on $[0, \infty)$.

(c) (4 points) Show that if $a > 0$, the sequence $\{f_n\}_{n\ge1}$ converges uniformly to the zero function on $[a, \infty)$.
(Tip: Use the Taylor series expansion of e^{nx} about $x = 0$ to show that $e^{nx} \ge \frac{n^3x^3}{3!}$ for all $x \ge 0$.)

Advanced Calculus II

3. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping given by

$$u = e^x \cos y, \quad v = e^x \sin y.$$

(a) (1 point) What is the range of f?

(b) (2 points) What are the images, under f, of lines parallel to the coordinate axes?

(c) (1 point) Is f one-to-one on $\mathbb{R}^2$?

(d) (6 points) Does every point $(x, y) \in \mathbb{R}^2$ have a neighborhood U such that f, restricted to U, is one-to-one? Is there a contradiction with question (c) above? Give an explanation.

4. **(a)** (2 points) State Fubini's Theorem for a *continuous* function $f : A \times B \to \mathbb{R}$, where $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are closed rectangles.

(b) (4 points)] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function with continuous partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$. Use Fubini's Theorem to prove that

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y).$$

(c) (4 points) For a continuous function $f : [a, b] \to \mathbb{R}$, prove the inequality

$$\left(\int_a^b f(x)\, dx \right)^2 \leq (b - a) \int_a^b f^2(x)\, dx,$$

with equality only for $f(x) =$ constant, $x \in [a, b]$.
Hint: Consider an integral

$$I = \int_a^b \int_a^b (f(x) - f(y))^2\, dx\, dy.$$

Complex Analysis

5. Let f be the entire function defined by

$$f(z) := (z - i)^3 e^{z^2}.$$

(a) (4 points) Find the maximum modulus and the minimum modulus of $f(z)$ on the closed disk $|z - i| \leq 2$.

(b) (3 points) Evaluate the contour integral

$$\int_\Gamma \frac{dz}{f(z)},$$

where , is the positively oriented circle $|z - i| = 2$.

(c) (3 points) Find the Laurent expansion of

$$g(z) := \frac{e^{2iz}}{f(z)}$$

in powers of $z - i$ on the annulus $0 < |z - i| < \infty$. What is the *principal part* of this expansion, and what is the residue of g at $z = i$?

6. Let h be the function defined by

$$h(z) := e^z + 3z^2.$$

(a) (3 points) Use Rouche's Theorem to prove that h has no zeros in the annulus $1 < |z| < 2$.

(b) (3 points) If $f(z)$ is an entire function and if $\left|\frac{f(z)}{z}\right| \to 0$ as $z \to \infty$, show that $f(z)$ is a constant.
Hint: Make use of the Cauchy integral formulas and Taylor's Theorem.

(c) (4 points) Use contour integration to evaluate the real improper integral

$$\int_{-\infty}^{\infty} \frac{\cos ex}{x^2 + e^{-2}}\, dx.$$

Hint: Consider the function $\exp(iez)/(z^2 + e^{-2})$.