Qualifying Examination #2 **April 24, 1995**

If you have any difficulty with the wording of the following problems please contact the supervisor immediately. All persons responsible for these problems, in principle, will be accessible during the entire duration of the exam.

Complex Analysis

1. (a) (2 points) Find the residue of the following function at $\pi/3$,

$$\frac{1-\cos z}{2\sin z - \sqrt{3}}.$$

(b) (2 points) Is the residue of $f+g$ at $z=a$ equal to the residue of f plus the residue of g?

(c) (2 points) Is the residue of fg at $z=a$ equal to the product of the residues of f and g?

(d) (2 points) Compare the residue of f at a simple pole $z=a\neq 0$ with the residue of $zf(z^2)$ at $z=a^{1/2}$, where $a^{1/2}$ is the principal square root of a.

(e) (2 points) Show that if f is analytic at all points of the extended plane, then f is a constant.

2. (a) (2 points) Find the radius of convergence of the following power series

$$1+3z+2z^2+9z^3+\cdots+2^n z^{2n}+3^{n+1}z^{2n+1}+\cdots.$$

(b) (2 points) Find a formula for the coefficient of z^n in the Taylor series expansion of $e^z/(1-z)$ about the origin.

(c) (3 points) Show that the Taylor series expansion of $1/(z^2+1)$ about the point 1 is given by

$$\frac{1}{z^2+1}=\sum_{n=1}^{\infty}(-1)^{n-1}2^{n/2}\sin(n\pi/4)(z-1)^{n-1},$$

and state what you know about the region of validity of this representation.

(d) (3 points) Show that if $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 1, if $a_n\geq 0$, and if $\sum_{n=0}^{\infty} a_n$ diverges, then $f(x)\to\infty$ as $x\to 1$ along the radius $(0,1)$.

3. (a) (4 points) Show that for each $n \geq 1$,

$$\int_1^n \ln(x)\,dx \leq \sum_{k=2}^{n} \ln(k) \leq \int_1^n \ln(x)\,dx + \ln(n).$$

(b) (2 point) Evaluate $\int_1^n \ln(x)\,dx$ for each $n \geq 1$.

(c) (4 points) Deduce from (a) and (b) that for each $n \geq 1$,

$$e\left(\frac{n}{e}\right)^n \leq n! \leq e\left(\frac{n}{e}\right)^n n.$$

4. Let $g : [a,b] \to \mathbb{R}$ be a differentiable function on $[a,b]$, where the differentiability of g at the end points a and b is such that

$$g'(a) = \lim_{\substack{x \to a \\ x > a}} \frac{g(x) - g(a)}{x - a}; \quad g'(b) = \lim_{\substack{x \to b \\ x < b}} \frac{g(x) - g(b)}{x - b}.$$

(a) (2 points) Show that if $g'(a) > 0$, then there exists a $\delta > 0$ such that

$$g(a) < g(x) \quad \text{for all} \quad a < x < a + \delta.$$

(b) (2 points) Show that if $g'(b) < 0$, then there exists a $\delta > 0$ such that

$$g(b) < g(x) \quad \text{for all} \quad b - \delta < x < b.$$

(c) (4 points) Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on $[a,b]$, and let $f'(a) < k < f'(b)$. Show that there is at least one point c in (a,b) such that $f'(c) = k$. (Hint: Consider the function $g(x) = k(x - a) - f(x)$, and argue that g attains a maximum value on $[a,b]$ that occurs at some point c in (a,b).)

(d) (2 points) Let $h : [-1,1] \to \mathbb{R}$ be defined by

$$h(x) := 1 \qquad \text{for } x > 0,$$

$$h(x) := 0 \qquad \text{for } x = 0,$$

$$h(x) := -1 \qquad \text{for } x < 0.$$

Show that h is **not** the derivative on $[-1,1]$ of any function on the interval $[-1,1]$.

Advanced Calculus II

5. (a) (6 points) Let $\langle . , . \rangle$ denote the usual inner product on $\mathbb{R}^n$, let A denote a real $n \times n$ symmetric matrix, and define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \langle Ax, x \rangle$. Use the definition of the derivative to show that the derivative of f at $x \in \mathbb{R}^n$ is given by the linear transformation $y \mapsto 2\langle Ax, y \rangle$.

   (b) (4 points) Suppose $g : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable, that $g(x_0) = y_0$, and that the derivative of g at x_0 is given by the linear transformation $y \mapsto By$, where B denotes an $n \times n$ matrix. With f as in part (a), determine the derivative of $f \circ g$ at x_0.

6. (a) (5 points) State the Implicit Function Theorem for functions $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$.

   (b) (5 points) Suppose $g : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable, $g(0) = 0$, and that the derivative of g at $0 \in \mathbb{R}^n$ is nonsingular. Using your statement of the Implicit Function Theorem, show there is an open neighborhood U of the origin in $\mathbb{R}^n$ and a differentiable function $h : U \to \mathbb{R}^n$ such that $g \circ h$ is the identity on U.