Qualifying Examination II **October 7, 1996**

If you have difficulty with the wording of any of the following problems, please contact the supervisor immediately. All persons responsible for these problems will, in principle, be accessible during the entire duration of the exam.

Advanced Calculus I

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Assume that there exists a sequence $\{a_n\}$ of non-zero real numbers such that $\lim_{n\to\infty} a_n = 0$ and $f(a_n) = 0$ for all $n \geq 1$.

- **(a)** (2 points) Show that $f(0) = 0$.
- **(b)** (4 points) Show that there exists a sequence $\{c_n\}$ of non-zero real numbers such that $\lim_{n\to\infty} c_n = 0$ and $f'(c_n) = 0$ for all $n \geq 1$. (Hint: Use the Mean Value Theorem.)
- **(c)** (4 points) Show that $f''(0) = 0$.

2. **(a)** (2 points) Show that the power series $\sum_{n=1}^{\infty} n\, x^n$ is convergent for each x in $(-1, 1)$ and find its sum.

- **(b)** (3 points) Show that the power series $\sum_{n=1}^{\infty} n^2\, x^n$ is convergent for each x in $(-1, 1)$ and find its sum.
- **(c)** (2 points) Show that the following power series is convergent for each x in $(-1, 1)$ and find its sum.

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

- **(d)** (3 points) Show that the following power series is uniformly convergent on $[-1, 1]$ and find the function $f(x)$ to which it sums on $(-1, 1)$.

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Advanced Calculus II

3. **(a)** (1 point) Define what it means for a function $f : \mathbb{R}^m \to \mathbb{R}^n$ to be differentiable at a point $x_0 \in \mathbb{R}^m$.

(b) (4 points) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is such that the partial derivatives $D_1 f = \partial f / \partial x_1$ and $D_2 f = \partial f / \partial x_2$ exist and are continuous. Show that f is differentiable.

(c) (2 points) Is the following function differentiable at $(0,0)$? Give reasons.

$$f(x_1, x_2) = \begin{cases} \frac{|x_1 x_2|^{3/2}}{x_1^2 + x_2^2} & \text{if } (x_1, x_2) \neq (0,0) \\ 0 & \text{if } (x_1, x_2) = (0,0). \end{cases}$$

(d) (3 points) Show that the following function is not differentiable at $(0,0)$.

$$g(x_1, x_2) = \begin{cases} \frac{x_1 |x_2|}{\sqrt{x_1^2 + x_2^2}} & \text{if } (x_1, x_2) \neq (0,0) \\ 0 & \text{if } (x_1, x_2) = (0,0). \end{cases}$$

(Hint: Calculate the partial derivatives at $(0,0)$. Also find $h'(0)$, where $h(t) = g(t,t)$. How might this be a contradiction?)

4. Let $U \subset \mathbb{R}^m \times \mathbb{R}^n$ be an open subset. Let $f : U \to \mathbb{R}^m$ be a C^1 function. We say that f is a C^1 *function* if it is differentiable and its derivative is continuous. Suppose that there exists $(\tilde{y}, \tilde{x}) \in U$ such that $f(\tilde{y}, \tilde{x}) = 0$. Let A be the matrix of f' at $(\tilde{y}, \tilde{x})$. The implicit function theorem says that, *under certain conditions on* A, there exists a neighborhood $V \subset U$ of $(\tilde{y}, \tilde{x})$ and a neighborhood $W \subset \mathbb{R}^n$ of $\tilde{x}$ such that for any $x \in W$, there exists unique y with $(y, x) \in V$ such that $f(y, x) = 0$. Furthermore, there is a C^1 function $h : W \to \mathbb{R}^m$ such that $h(\tilde{x}) = \tilde{y}$, and $f(h(x), x) = 0$ for $x \in W$.

(a) (2 points) State precisely what the certain conditions on A are that guarantee that the conclusions of the implicit function theorem hold.

Now let $D = \{\, x \in \mathbb{R}^2 : x_1^2 + x_2^4 \leq 1 \,\}$, and for $x \in \mathbb{R}^2 \setminus \{0\}$, let

$$\| x \| = \inf \left\{ \lambda > 0 : \frac{x}{\lambda} \in D \right\}.$$

The ultimate objective in the remaining parts of this problem is to show that the map

$$x \mapsto \| x \| \quad (x \in \mathbb{R}^2 \setminus \{0\})$$

is a C^1 function. In order to do this, let us define a function $f : (0, \infty) \times (\mathbb{R}^2 \setminus \{0\}) \to \mathbb{R}$ by

$$f(\lambda, x) = \left(\frac{x_1}{\lambda} \right)^2 + \left(\frac{x_2}{\lambda} \right)^4 - 1.$$

(b) (3 points) Show that for any $x \in \mathbb{R}^2 \setminus \{0\}$ and any $\lambda > 0$ that $f(\lambda, x) = 0$ if and only if $\lambda = \| x \|$.

(c) (3 points) Show that for any $\tilde{x} \in \mathbb{R}^2 \setminus \{0\}$ there exists a neighborhood $W \subset \mathbb{R}^2$ of $\tilde{x}$ and a C^1 function $h : W \to (0, \infty)$ such that $f(h(x), x) = 0$ for $x \in W$.

(d) (2 points) Conclude that the map $x \mapsto \| x \|$, $(x \in \mathbb{R}^2 \setminus \{0\})$ is a C^1 function.

Complex Analysis

5. You may assume the following result when working this problem: If $f(z)$ is analytic on a domain (open connected set) $D \subset \mathbb{C}$ and if $f'(z) = 0$ for all $z \in D$, then f is constant on D.

(a) (2 points) Prove that, if $f(z)$ and $\overline{f(z)}$ are both analytic on a domain $D \subset \mathbb{C}$, then $f(z)$ is constant on D. (Hint: Make use of the Cauchy-Riemann equations.)

(b) (2 points) Prove that, if $f(z)$ is analytic on a domain $D \subset \mathbb{C}$ and $|f(z)|$ is constant on D, then $f(z)$ is constant on D.

(c) (3 points) Let $f(z)$ be an entire function, and assume that $f(z)$ is real valued for all z such that $|z| = 1$. Show that $f(z)$ is constant on $\mathbb{C}$. (Hint: Apply max/min principles to the function $g(z) = \exp(-i\, f(z))$.)

(d) (3 points) Let $f(z)$ be an entire function, with $|f(z)| = 1$ for all z such that $|z| = 1$. Show that $f(z) = k\, z^n$ for some constant k ($|k| = 1$) and some nonnegative integer n. (Hint: Consider the function

$$h(z) = f(z) / \prod_{j=1}^{n} \left[\frac{z - a_j}{1 - \overline{a_j}\, z} \right],$$

where the a_j are the zeros of f each repeated as often as its multiplicity.)

6. (a) (4 points) Show how contour integration can be used to evaluate the following real integral.

$$\int_{-\pi}^{\pi} \frac{d\theta}{5 + 4\sin\theta}$$

(b) (3 points) Use contour integration to evaluate the following improper integral for each positive integer $n \geq 2$.

$$\int_0^\infty \frac{x^{n-2}}{1 + x^n}\, dx$$

(Hint: Consider the closed contour composed of the ray from 0 to R (> 1) followed by the circular arc $R\, e^{i\theta}$ from $\theta = 0$ to $\theta = 2\pi/n$ followed by the ray from $R\, e^{2\pi i/n}$ to 0.)

(c) (3 points) Let Γ be the circle with center 0 and radius 1 oriented in the counter-clockwise manner. For what positive integers m is

$$\int_\Gamma z^{m-1} e^{-1/z} \cos(1/z)\, dz = 0 \ ? \qquad \left(\text{Hint: } \cos w = \frac{e^{iw} + e^{-iw}}{2} \right)$$