Qualifying Examination II **April 29, 1996**

If you have difficulty with the wording of any of the following problems, please contact the supervisor immediately. All persons responsible for these problems will, in principle, be accessible during the entire duration of the exam.

Advanced Calculus I

1. Let $f : (a, b) \to \mathbb{R}$ be a given function, where $a, b \in \mathbb{R}$ and $a < b$.

- **(a)** (2 points) Prove that f is uniformly continuous on (a, b) if and only if for every pair of sequences $\{x_n\}$, $\{y_n\}$ in (a, b) such that $|x_n - y_n| \to 0$ as $n \to \infty$, we have that $|f(x_n) - f(y_n)| \to 0$ as $n \to \infty$.
- **(b)** (2 points) Show that if f is uniformly continuous on (a, b), then f takes Cauchy sequences in (a, b) into convergent sequences in $\mathbb{R}$.
- **(c)** (3 points) Show that if f is uniformly continuous on (a, b), then f can be extended to a continuous function on $[a, b]$.
- **(d)** (3 points) Show that if $f'(x)$ exists for all x in (a, b) and if f' is bounded on (a, b), then f can be extended to a uniformly continuous function on $[a, b]$.
  (Hint: Show that there exists a constant $M \geq 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in (a, b)$.)

2. (a) (3 points) Suppose $f_1, f_2, f_3, \ldots$ are continuous functions on a closed bounded interval $[a, b]$, and suppose that the sequence $\{f_n\}$ converges uniformly to a limit function f on $[a, b]$. Show that

$$\lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

- **(b)** (4 points) Show that the sequence $\{f_n\}$ defined by $f_n(x) = n^2 x^n (1 - x)$ converges pointwise to $f = 0$ on $[0, 1]$.
- **(c)** (3 points) Show that the sequence $\{f_n\}$ defined in (b) above does not converge uniformly to $f = 0$ on $[0, 1]$.

Advanced Calculus II

3. (*Implicit Function Theorem*) Let $\mathbf{f}$ be a continuously differentiable map of an open set $E \subset \mathbb{R}^{n+m}$ to $\mathbb{R}^n$. Assume $(\mathbf{a}, \mathbf{b}) \in E$, $\mathbf{f}(\mathbf{a}, \mathbf{b}) = 0$ and the differential $A = D\mathbf{f}(\mathbf{a}, \mathbf{b})$ satisfies

$$A(\mathbf{h}, \mathbf{0}) = \mathbf{0} \text{ is equivalent to } \mathbf{h} = \mathbf{0}. \tag{1}$$

(a) (3 points) Formulate the Implicit Function Theorem.

(b) (1 point) Which theorem provides the main step in the proof of the Implicit Function Theorem?

(c) (2 points) For $m = 1$, $n = 2$, consider

$$\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (xy, x + y). \tag{2}$$

Compute $A = D\mathbf{f}$ at $(x, y, z) \in \mathbb{R}^3$, where $\mathbf{f}$ is defined by (2).

(d) (1 point) Does the Implicit Function Theorem apply to $\mathbf{f}$ defined by (2) when $\mathbf{a} = (0, 0)$ and $\mathbf{b} = 0$? Explain your yes or no answer.

(d) (3 points) (*Implicit Differentiation*) Find du and dv if

$$u + v = x + y, \qquad y \sin u - x \sin v = 0.$$

4. (*Extrema*) (a) (2 points) Give the definition of the *local* maximum of a function $f : E \to \mathbb{R}$, $E \subset \mathbb{R}^n$ at a point $\mathbf{a} \in E$. Under which conditions on E is the *global* maximum of f attained on E?

(b) (1 point) Give a necessary condition in terms of its differential for a differentiable function $f : E \to \mathbb{R}$, $E \subset \mathbb{R}^n$, to have a local maximum at a point $\mathbf{a} \in E$.

(c) (2 points) Formulate a sufficient condition in terms of partial derivatives for a function $f : E \to \mathbb{R}$, $E \subset \mathbb{R}^2$, to have a local maximum at a point $\mathbf{a} = (x, y) \in E$.

(d) (2 points) Let $f(x, y) = x^3 + y^3 - 3xy$. Find the stationary points and the local extrema of this function.

(e) (3 points) Assume a function $f : \mathbb{R}^2 \to \mathbb{R}$ has a minimum at a point $\mathbf{a} = (x_0, y_0) \in \mathbb{R}^2$ along each line going through this point. Does this imply that f has a local minimum at $\mathbf{a}$?
(Hint: Take $f(x, y) = (x - y^2)(2x - y^2)$ and $\mathbf{a} = (0, 0)$.)

Complex Analysis

5. (a) (3 points) With respect to isolated singularities of a complex function, what does Riemann's Theorem tell us about removable singular points, and what does the Casorati-Weierstrass Theorem tell us about essential singular points? State a theorem about poles of a complex function.

(b) (4 points) Classify the singularity and determine the residue at the indicated point for each of the following functions.

$$(1)\ \frac{z\sin \pi z}{(z-1)^2};\ 1, \qquad (2)\ \frac{z-\pi}{\tan z};\ \pi, \qquad (3)\ e^{\sin(2/z)};\ 0$$

(c) (3 points) Use contour integration to evaluate the following improper integral.

$$\int_{-\infty}^{\infty} \frac{dx}{x^2-2x+5}$$

6. (a) (2 points) Find the image of the circle $C:\ \{\, z : |z-4-3i| = 5 \,\}$ under the mapping

$$w = \frac{1}{z}.$$

(b) (3 points) Suppose that $|f(z)| \le M$ on the circumference of a square whose side is L, and let z_0 be the center of the square. If f is analytic in the square, show that $|f'(z_0)| \le 8M/(\pi L)$.

(c) (3 points) Let f be an entire function. State Liouville's Theorem in terms of f and show that the conclusion of this theorem still holds if the boundedness of f is replaced by the boundedness of

$$\int_0^{2\pi} |f(re^{i\theta})|\, d\theta, \quad r > 0.$$

(d) (2 points) State a version of Morera's Theorem and show how it can be used to prove that the function f defined by

$$f(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

is analytic in the unit disk $D = \{\, z : |z| < 1 \,\}$.