Qualifying Examination II **August 22, 1997**

If you have difficulty with the wording of any of the following problems, please contact the supervisor immediately. All persons responsible for these problems will, in principle, be accessible during the entire duration of the exam.

Advanced Calculus I

1. Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable at $\mathbb{R}$.

(a) (4 points) Show that for any $\alpha,\beta\in\mathbb{R}$ we have

$$\lim_{h\to 0}\frac{f(x-\alpha h)-f(x-\beta h)}{h}=(\beta-\alpha)\,f'(x).$$

(b) (6 points) If f' exists and is bounded on a subset A of $\mathbb{R}$, prove that f is uniformly continuous on A.

2. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two infinite series of real numbers such that $a_n>0$ and $b_n>0$ for all $n>1$.

(a) (3 points) Show that if

$$\lim_{n\to\infty}\frac{a_n}{b_n}=L,\qquad 0<L<\infty,$$

then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

(b) (2 points) Given that

$$a_n=n^{\frac{1}{n}}-1 \text{ and } b_n=\frac{\ln n}{n},\qquad n\ge 1,$$

find

$$\lim_{n\to\infty}\frac{a_n}{b_n}.$$

(c) (3 points) Show that the series

$$\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$$

is convergent.

(d) (2 points) Show that the series

$$\sum_{n=1}^{\infty}\frac{n^{\frac{1}{n}}-1}{n}$$

is convergent.

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Advanced Calulus II

3. Suppose that $E \subset \mathbb{R}^n$.

(a) (2 points) Define the interior of E.

(b) (2 points) Define what it means for E to be convex.

(c) (6 points) Show that if E is convex, then the interior of E is convex.

4. (a) (2 points) State clearly the mean value theorem concerning a function $f : [a, b] \to \mathbb{R}$.

State whether the statements in the following parts are true or false, and give reasons for your answers. (Points will only be obtained for correct reasons. The only version of the mean value theorem that you may quote is the one you just stated.)

(b) (3 points) If $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable, then for any $x, y \in \mathbb{R}^m$ there exists $0 < t < 1$ such that

$$f(x) - f(y) = Df(tx + (1 - t)y)(x - y).$$

(Hint: Consider the function $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(x) = (\cos x, \sin x)$.)

(c) (5 points) If $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable and $Df(x) = 0$ for all $x \in \mathbb{R}^m$, then f is constant. (Hint: For $x, y \in \mathbb{R}^m$, consider the function $g : [0, 1] \to \mathbb{R}$ given by

$$g(t) = (f(x) - f(y)) \cdot f(tx + (1 - t)y).)$$

Complex Analysis

5. Let $Q = \{ z : \Re(z) \geq 0 \text{ and } \Im(z) \geq 0 \}$ be the first quadrant and f be the transformation defined by

$$f(z) = i\left(\frac{z^2 - i}{z^2 + i}\right).$$

(a) (3 points) What is the image of Q under the transformation f?

(b) (2 points) Is f analytic in an open set containing Q?

(c) (2 points) Does the transformation f have a fixed point in Q?

(d) (3 points) What are the isolated singularities of f and their corresponding residues?

6. (a) (3 points) If α is a complex constant, show that the principal value of $(1 + z)^\alpha$ is bounded on the unit disk $|z| < 1$ if and only if $\Re(\alpha) \geq 0$.

(b) (3 points) For what values of $z = x + iy$ is $\sinh \bar{z}$ differentiable?

(c) (4 points) Carefully show how contour integration can be used to evaluate the real improper integral

$$\int_0^\infty \frac{2x \sin x}{x^2 + 1}\, dx.$$