

If you have difficulty with the wording of any of the following problems, please contact the supervisor immediately. All persons responsible for these problems will, in principle, be accessible during the entire duration of the exam.

Advanced Calculus I

1. (a) (2 points) State the Mean Value Theorem for derivatives.

(b) (8 points) Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and assume the second derivative f'' exists on (a, b) . Suppose the graph of f and the line segment joining the points $(a, f(a))$ and $(b, f(b))$ intersect at a point $(x_0, f(x_0))$, where $a < x_0 < b$. Show that there exists c in (a, b) such that $f''(c) = 0$.

2. Let

$$f_n(x) := \frac{\sin(nx)}{nx}, \quad x \in (0, \pi].$$

(a) (3 points) Show that the sequence $\{f_n\}$ **converges pointwise** to the zero function on $(0, \pi]$.

(b) (3 points) Show that the sequence $\{f_n\}$ **does not converge uniformly** to the zero function on $(0, \pi]$.

(c) (4 points) If $a \in (0, \pi]$, show that

$$\lim_{n \rightarrow \infty} \int_a^\pi f_n(x) dx = 0.$$

Advanced Calculus II

3. (a) (3 points) State properly the inverse function theorem. Now let us consider the function

$$f(x, y) = \left(\frac{x^2 + y^2}{2}, xy \right).$$

(b) (3 points) Use the inverse function theorem to show that if $|x_0| \neq |y_0|$, then there exist neighborhoods V of (x_0, y_0) and W of $(u_0, v_0) = f(x_0, y_0)$ such that $f : V \rightarrow W$ is invertible.

(c) (1 point) Continuing with part (b), without explicitly finding f^{-1} , find the derivative of f^{-1} at (u_0, v_0) in terms of (x_0, y_0) .

(d) (2 points) Show that if f is defined with domain $\{(x, y) : |x| \neq |y|\}$, then the range of f is the set $\{(u, v) : u > |v|\}$.

(e) (1 point) Is the function $f : \{(x, y) : |x| \neq |y|\} \rightarrow \{(u, v) : u > |v|\}$ invertible?

4. (a) (3 points) State properly the multidimensional version of Taylor's Theorem, that is, given $f : \mathbb{R}^n \rightarrow \mathbb{R}$ that is N -times continuously differentiable, express $f(a + h)$ in terms of partial derivatives of f at x with a remainder term expressed in terms of N -th partial derivatives of f .

- (b) (3 points) Show that if $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is three times continuously differentiable, then

$$\lim_{r \rightarrow 0^+} \frac{2}{\pi r^2} \int_0^{2\pi} (f(x + r \cos \theta, y + r \sin \theta) - f(x, y)) d\theta = f_{xx}(x, y) + f_{yy}(x, y).$$

- (c) (4 points) Show that if $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is three times continuously differentiable and is such that $f(0, 0) \leq f(x, y)$ for all $(x, y) \in \mathbb{R}^2$, then for all $h, k \in \mathbb{R}$ we have that

$$f_{xx}(0, 0) h^2 + 2f_{xy}(0, 0) hk + f_{yy}(0, 0) k^2 \geq 0.$$

Complex Analysis

5. Let the function f be defined on the disk $D = \{z : |z| < 4\}$ by $f(z) := z^2 / \sin z$.

- (a) (2 points) Determine and classify the singularities of f in D .
- (b) (3 points) If f has any isolated singularities in D , determine the residue of f at each such point.
- (c) (2 points) Evaluate $\int_{\gamma} f(z) dz$, where γ is the counterclockwise oriented circle with center at the origin and radius 4.
- (d) (3 points) Now let the function g be defined by $g(z) := z - \sin z$. Assuming that the only zero of g in the closed unit disk is $z = 0$, evaluate

$$\int_{\Gamma} \frac{dz}{z - \sin z},$$

where Γ is the counterclockwise oriented unit circle.

6. (a) (3 points) Why is the function h defined by $h(z) := e^{\bar{z}}$ differentiable nowhere?
- (b) (3 points) Find the Taylor series expansion of $\arctan z$ about $z = 0$. What is the radius of convergence of this expansion?
- (c) (2 points) Suppose f is analytic in a domain containing the closed unit disk $U = \{z : |z| \leq 1\}$. Suppose also that $f(0) = 0$ and $|f(z)| \geq 1$ for $|z| = 1$. Show that if α is a complex number such that $|\alpha| < 1/2$, then the equation $f(z) = \alpha$ has a solution in U . (Hint: Consider $1/(f(z) - \alpha)$.)
- (d) (2 points) Prove that the only root of the equation $z - \sin z = 0$ that lies in the closed unit disk is $z = 0$.