HEREDITY OF WHITTAKER MODELS ON THE METAPLECTIC GROUP

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In this paper, Rodier's theorem on the heredity of Whittaker models is generalized to non-algebraic setting of the n-fold metaplectic cover of the general linear group $GL_r(\mathbb{F})$, where $\mathbb{F}$ is nonarchimedean local field containing the n-th roots of unity.

§1. Introduction.

Let $\mathbb{F}$ be a nonarchimedean local field, let G be the general linear group $GL_r(\mathbb{F})$ for some positive integer r, and let P be a standard parabolic subgroup of G with Levi component M. Given an admissible representation π_M of M, extend π_M to a representation π_P of P by letting the unipotent radical of P act trivially, and let π_G be the normalized full-induced representation $\mathrm{Ind}(P, G; \pi_P)$. Then by a well-known result of F. Rodier, there exists a correspondence between the Whittaker models of the induced representation π_G and the Whittaker models of the inducing representation π_M (cf. Theorem 2 of [4]).

In this paper, Rodier's theorem on the "heredity" of Whittaker models is extended to the *non-algebraic* setting of the n-fold metaplectic cover $\widetilde{G}$ of G, where n is a positive integer such that $\mathbb{F}$ contains all of the n-th roots of unity. The main result is stated as a theorem in §2. In order to illustrate the situation, consider the example of a representation of $\widetilde{G}$ induced from the metaplectic preimage $\widetilde{B}$ of the standard Borel subgroup B of G. Since the Levi component T of B is a (maximal) torus in G, its metaplectic preimage $\widetilde{T}$ is a *Heisenberg group*. Consequently, the dimension of any irreducible representation π_T of $\widetilde{T}$ is equal to the index $[\widetilde{T} : \widetilde{T}_*]$, where $\widetilde{T}_*$ is an arbitrary maximal abelian subgroup of $\widetilde{T}$. In this example, *every* linear functional on the space of π_T is a Whittaker functional, hence the inducing representation π_T has precisely $[\widetilde{T} : \widetilde{T}_*]$ distinct Whittaker models. Now extend π_T to a representation π_B of $\widetilde{B}$ (see §2 below), and let π_G be the normalized, full-induced representation $\mathrm{Ind}(\widetilde{B}, \widetilde{G}; \pi_B)$ of $\widetilde{G}$. By Lemma I.3.2 of [3], it follows that π_G also has $[\widetilde{T} : \widetilde{T}_*]$ distinct Whittaker models, thus Rodier's theorem evidently extends to this example.

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While the main techniques of proof employed in this paper are contained in [4], it is unclear *a priori* that those techniques carry over to the metaplectic group. Here the situation is clarified by a close examination of various aspects of Rodier's proof.

The results of this paper will be relevant to the generalization of F. Shahidi's theory of local coefficients $\big($cf. [5]$\big)$ to the metaplectic setting, to the construction of certain non-principal theta functions $\big($cf. [3]$\big)$, and to the eventual classification of metaplectic representations.

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§2. The Metaplectic Group, Whittaker Models, and Heredity.

Let n and r be fixed positive integers. Let $\mathbb{F}$ be a nonarchimedean local field, and let μ_n denote the group of n-th roots of unity in $\mathbb{F}$. *We will assume that μ_n has cardinality n.*

Let $\widetilde{G}$ denote the *n-fold metaplectic cover* of $G := GL_r(\mathbb{F})$ $\big($cf. §0.I of [3]$\big)$. As a set, $\widetilde{G} = G \times \mu_n$, with multiplication defined by:

$$(g,\zeta)\cdot(g',\zeta') = \big(gg', \zeta\zeta'\sigma(g,g')\big), \qquad \forall\ g, g' \in G,\ \zeta, \zeta' \in \mu_n.$$

Here $\sigma : G \times G \to \mu_n$ is the *Matsumoto 2-cocycle* in $\mathrm{H}^2(G;\mu_n)$. Let $\mathbf{s} : G \to \widetilde{G}$ be the preferred section $g \mapsto (g,1)$, and let $\mathbf{p} : \widetilde{G} \to G$ be the canonical projection $(g,\zeta) \mapsto g$.

Let N be the unipotent radical of the standard Borel subgroup B of G, and let $N^{\mathbf{s}} := \mathbf{s}(N)$. Since $\sigma\big|_{N\times N} = 1$, $\mathbf{s} : N \to N^{\mathbf{s}}$ is a group isomorphism. Once and for all, let ψ be a fixed *principal* character of N $\big($cf. §2 of [4]$\big)$. Then for any positive simple root α of G, the restriction of ψ to the unipotent root group N_α is *nontrivial*. Let ψ^* denote the corresponding character $\psi \circ \mathbf{p} = \psi \circ \mathbf{s}^{-1}$ of $N^{\mathbf{s}}$, and let $\bar{\psi}^*$ be the character obtained from ψ^* by complex conjugation.

Let W be the Weyl group of permutation matrices in G, and let $W^{\mathbf{s}} := \mathbf{s}(W)$. If the *$n$-th order Hilbert symbol* $(\cdot,\cdot)_{\mathbb{F}} : \mathbb{F}^\times \times \mathbb{F}^\times \to \mu_n$ satisfies the relation $(-1,-1)_{\mathbb{F}} = 1$, then $\mathbf{s} : W \to W^{\mathbf{s}}$ is a group isomorphism, but we will proceed without this assumption.

Let $M \subseteq G$ be the Levi component of an arbitrary standard parabolic subgroup P of G. Then $P = MU$, and $M \cap U = \{e\}$, where $U \subseteq N$ is the unipotent radical of P, and e is the identity of G. Let $\widetilde{P}$, $\widetilde{M}$, and $U^{\mathbf{s}}$ denote the subgroups $\mathbf{p}^{-1}(P)$, $\mathbf{p}^{-1}(M)$, and $\mathbf{s}(U)$ of $\widetilde{G}$, respectively. Then $\widetilde{P} = \widetilde{M}U^{\mathbf{s}}$, and $\widetilde{M} \cap U^{\mathbf{s}} = \{\tilde{e}\}$, where $\tilde{e}$ is the identity of $\widetilde{G}$.

Let $W_{M} := W \cap M$. As in §1 of [2], let:

$$[W/W_{M}] := \big\{ w \in W \;\big|\; w \text{ is of minimal length in } wW_{M} \in W/W_{M} \big\},$$

and let w_{M} denote the longest element of $[W/W_{M}]$. Let $N_{M} := N \cap M$, and let $N^{\mathbf{s}}_{M} := \mathbf{s}(N_{M})$. Since $w_{M} N_{M} w_{M}^{-1} \subseteq N$, we can a define a character $\psi^{*}_{M} : N^{\mathbf{s}}_{M} \to \mathbb{C}^{\times}$ by:

$$\psi^{*}_{M}(n) := \psi\big(w_{M}\, \mathbf{p}(n)\, w_{M}^{-1}\big), \qquad \forall\, n \in N^{\mathbf{s}}_{M}.$$

Let $\bar{\psi}^{*}_{M}$ be the character obtained from ψ^{*}_{M} by complex conjugation. In particular, we have that $N^{\mathbf{s}}_{G} = N^{\mathbf{s}}$, and $w_{G} = e$, hence $\psi^{*}_{G} = \psi^{*}$, and $\bar{\psi}^{*}_{G} = \bar{\psi}^{*}$.

Let Π^{ψ}_{M} denote the *full-induced* representation $\mathrm{Ind}(N^{\mathbf{s}}_{M}, \widetilde{M}; \psi^{*}_{M})$ of $\widetilde{M}$. The space $\mathcal{W}^{\psi}_{M}$ of Π^{ψ}_{M} consists of the locally-constant functions $f : \widetilde{M} \to \mathbb{C}$ that satisfy $f(ng) = \psi^{*}_{M}(n)\, f(g)$ for all $n \in N^{\mathbf{s}}_{M}$ and $g \in \widetilde{M}$, and $\widetilde{M}$ acts on $\mathcal{W}^{\psi}_{M}$ by right translation. Similarly, let ${}^{\circ}\Pi^{\bar{\psi}}_{M}$ denote the *compactly-induced* representation $\mathrm{Ind}^{\circ}(N^{\mathbf{s}}_{M}, \widetilde{M}; \bar{\psi}^{*}_{M})$ of $\widetilde{M}$. The space ${}^{\circ}\mathcal{W}^{\bar{\psi}}_{M}$ of ${}^{\circ}\Pi^{\bar{\psi}}_{M}$ consists of the locally-constant functions $f : \widetilde{M} \to \mathbb{C}$ that are compactly supported modulo $N^{\mathbf{s}}_{M}$ and satisfy $f(ng) = \bar{\psi}^{*}_{M}(n)\, f(g)$ for all $n \in N^{\mathbf{s}}_{M}$ and $g \in \widetilde{M}$. Then $\widetilde{M}$ also acts on ${}^{\circ}\mathcal{W}^{\bar{\psi}}_{M}$ by right translation. By Proposition 2.25(c) of [1], ${}^{\circ}\Pi^{\bar{\psi}}_{M}$ is the *contragredient* of the representation Π^{ψ}_{M}.

Let π_{M} be a smooth representation of $\widetilde{M}$. A subspace $\mathcal{W}$ of $\mathcal{W}^{\psi}_{M}$ is said to be a ψ^{*}_{M}*-Whittaker model for* π_{M} if $\mathcal{W}$ is $\widetilde{M}$-invariant, and the restriction of Π^{ψ}_{M} to $\mathcal{W}$ is a representation that is equivalent to π_{M}. In other words, $\mathcal{W}$ is the image of an *injective* element of $\mathrm{Hom}_{\widetilde{M}}(\pi_{M}, \Pi^{\psi}_{M})$.

The following theorem is a generalization to the metaplectic group of Rodier's theorem on the heredity of Whittaker models $\big($cf. Theorem 2 of [4]$\big)$.

Theorem. *Let $\pi_{_M}$ be an admissible representation of $\widetilde{M}$. Extend $\pi_{_M}$ to a representation $\pi_{_P}$ of $\widetilde{P}$ by letting $U^{\mathbf{s}}$ act trivially, and let $\pi_{_G}$ be the normalized induced representation* $\mathrm{Ind}(\widetilde{P},\widetilde{G};\pi_{_P})$ *of $\widetilde{G}$. Then* $\mathrm{Hom}_{\widetilde{G}}(\pi_{_G},\Pi^{\psi}_{_G}) \cong \mathrm{Hom}_{\widetilde{M}}(\pi_{_M},\Pi^{\psi}_{_M})$.

Proof: A topological space X is an *l-space* if it is Hausdorff, locally-compact, and zero-dimensional (cf. §1.1 of [1]). For any l-space X and any complex vector space V, let $\mathcal{S}(X;V)$ denote the space of locally-constant, compactly-supported functions from X to V, and let $\mathcal{D}(X;V)$ be the linear dual of $\mathcal{S}(X;V)$. When $V=\mathbb{C}$, we will simply write $\mathcal{S}(X)$ and $\mathcal{D}(X)$, respectively. Any element of $\mathcal{D}(X;V)$ [resp. $\mathcal{D}(X)$] is called a *V-distribution* [resp. *distribution*].

Let $\mathcal{V}$ denote the space of $\pi_{_M}$. For any l-subspace X of $\widetilde{G}$ such that $N^{\mathbf{s}}X\widetilde{P}=X$, let $\mathcal{D}^1(X)$ denote the space of $\mathcal{V}$-distributions $D\in\mathcal{D}(X;\mathcal{V})$ that satisfy:

$$D\big(\lambda^1_{_N}(n)\rho^1_{_P}(p)\varphi\big)=\bar{\psi}^*(n)\,\delta(p)^{-1/2}\,D\big(\pi_{_P}(p^{-1})\circ\varphi\big),\qquad \forall\; n\in N^{\mathbf{s}},\; p\in\widetilde{P},\; \varphi\in\mathcal{S}(X;\mathcal{V}).$$

Here $\lambda^1_{_N}:N^{\mathbf{s}}\to\mathrm{Aut}\big(\mathcal{S}(X;\mathcal{V})\big)$ and $\rho^1_{_P}:\widetilde{P}\to\mathrm{Aut}\big(\mathcal{S}(X;\mathcal{V})\big)$ are the representations defined in the usual way by left and right translation, respectively, and $\delta:\widetilde{P}\to\mathbb{C}^{\times}$ is the modular character of $\widetilde{P}$.

By a theorem of F. Bruhat (cf. Theorem 4 of [4]), $\mathcal{D}^1(\widetilde{G})$ is isomorphic to the space $\mathrm{Bil}_{\widetilde{G}}({}^{\circ}\Pi^{\bar{\psi}}_{_G},\pi_{_G})$ of $\widetilde{G}$-invariant bilinear forms on ${}^{\circ}\mathcal{W}^{\bar{\psi}}_{_G}\times\mathcal{V}$ (i.e., *intertwining forms* in the sense of §1 of [4]). Here we have used the fact that $\widetilde{G}=\widetilde{P}K$ for some compact, open subset K of $\widetilde{G}$, and that $\bar{\psi}^*_{_G}=\bar{\psi}^*$. Since ${}^{\circ}\Pi^{\bar{\psi}}_{_G}$ is the contragredient of the representation $\Pi^{\psi}_{_G}$, it can also be shown that $\mathrm{Hom}_{\widetilde{G}}(\pi_{_G},\Pi^{\psi}_{_G})\cong\mathrm{Bil}_{\widetilde{G}}({}^{\circ}\Pi^{\bar{\psi}}_{_G},\pi_{_G})$. Hence, $\mathcal{D}^1(\widetilde{G})\cong\mathrm{Hom}_{\widetilde{G}}(\pi_{_G},\Pi^{\psi}_{_G})$, and it remains to show that $\mathcal{D}^1(\widetilde{G})\cong\mathrm{Hom}_{\widetilde{M}}(\pi_{_M},\Pi^{\psi}_{_M})$.

For every $w\in[W/W_{_M}]$, let $\tilde{w}:=\mathbf{s}(w)$. Starting from the Bruhat decomposition for G, one can show that $\widetilde{G}=\coprod_w N^{\mathbf{s}}\tilde{w}\widetilde{P}$, where the disjoint union is taken over all $w\in[W/W_{_M}]$ (cf. §1 of [2]). In order to describe $\mathcal{D}^1(\widetilde{G})$, it will suffice to study each space $\mathcal{D}^1(N^{\mathbf{s}}\tilde{w}\widetilde{P})$ separately. Thus, let w be a fixed element of $[W/W_{_M}]$. For every $\varphi\in\mathcal{S}(N^{\mathbf{s}}\times\widetilde{P};\mathcal{V})$, let $\hat{\varphi}\in\mathcal{S}(N^{\mathbf{s}}\tilde{w}\widetilde{P};\mathcal{V})$ be defined by:

$$\hat{\varphi}(n\tilde{w}p):=\int_{N^{\mathbf{s}}\cap\tilde{w}\widetilde{P}\tilde{w}^{-1}}\varphi(nn_{\circ},\tilde{w}^{-1}n_{\circ}^{-1}\tilde{w}p)\,dn_{\circ},\qquad \forall\; n\in N^{\mathbf{s}},\; p\in\widetilde{P},$$

where $dn_\circ$ is a Haar measure for $N^{\mathbf{s}} \cap \tilde{w}\widetilde{P}\tilde{w}^{-1}$. The map $\varphi \mapsto \hat{\varphi}$ is surjective, hence by duality it follows that $\mathcal{D}^1(N^{\mathbf{s}}\tilde{w}\widetilde{P})$ is isomorphic to the space $\mathcal{D}^2$ of $\mathcal{V}$-distributions $D \in \mathcal{D}(N^{\mathbf{s}}\times\widetilde{P};\mathcal{V})$ that satisfy:

$$\begin{aligned} D\big(\lambda^2_{N}(n)\rho^2_{P}(p)\varphi\big) &= \bar{\psi}^*(n)\,\delta(p)^{-1/2}\,D\big(\pi_P(p^{-1})\circ\varphi\big), && \forall\, n \in N^{\mathbf{s}},\ p \in \widetilde{P},\\ D\big(\rho^2_{N}(n_\circ)\varphi\big) &= D\big(\lambda^2_{P}(\tilde{w}^{-1}n_\circ\tilde{w})\varphi\big), && \forall\, n_\circ \in N^{\mathbf{s}} \cap \tilde{w}\widetilde{P}\tilde{w}^{-1}, \end{aligned}$$

for all $\varphi \in \mathcal{S}(N^{\mathbf{s}}\times\widetilde{P};\mathcal{V})$. Here λ^2_{N} and λ^2_{P} are representations defined by left translation:

$$\lambda^2_{N} : N^{\mathbf{s}} \to \operatorname{Aut}\big(\mathcal{S}(N^{\mathbf{s}}\times\widetilde{P};\mathcal{V})\big), \qquad \lambda^2_{P} : \widetilde{P} \to \operatorname{Aut}\big(\mathcal{S}(N^{\mathbf{s}}\times\widetilde{P};\mathcal{V})\big),$$

and ρ^2_{N} and ρ^2_{P} are representations defined by right translation:

$$\rho^2_{N} : N^{\mathbf{s}} \to \operatorname{Aut}\big(\mathcal{S}(N^{\mathbf{s}}\times\widetilde{P};\mathcal{V})\big), \qquad \rho^2_{P} : \widetilde{P} \to \operatorname{Aut}\big(\mathcal{S}(N^{\mathbf{s}}\times\widetilde{P};\mathcal{V})\big).$$

Next, we identify $\mathcal{S}(N^{\mathbf{s}}\times\widetilde{P};\mathcal{V})$ with $\mathcal{S}(N^{\mathbf{s}}) \otimes \mathcal{S}(\widetilde{P};\mathcal{V})$ in the usual way, that is, for every $\varphi_{N} \in \mathcal{S}(N^{\mathbf{s}})$ and $\varphi_{P} \in \mathcal{S}(\widetilde{P};\mathcal{V})$, let $\varphi_{N}\otimes\varphi_{P} \in \mathcal{S}(N^{\mathbf{s}}\times\widetilde{P};\mathcal{V})$ be defined by:

$$(\varphi_{N}\otimes\varphi_{P})(n,p) := \varphi_{N}(n)\,\varphi_{P}(p), \qquad \forall\, n \in N^{\mathbf{s}},\ p \in \widetilde{P}.$$

Let λ^3_{N} and λ^3_{P} be the representations defined by left translation:

$$\lambda^3_{N} : N^{\mathbf{s}} \to \operatorname{Aut}\big(\mathcal{S}(N^{\mathbf{s}})\big), \qquad \lambda^3_{P} : \widetilde{P} \to \operatorname{Aut}\big(\mathcal{S}(\widetilde{P};\mathcal{V})\big),$$

and let ρ^3_{N} and ρ^3_{P} be the representations defined by right translation:

$$\rho^3_{N} : N^{\mathbf{s}} \to \operatorname{Aut}\big(\mathcal{S}(N^{\mathbf{s}})\big), \qquad \rho^3_{P} : \widetilde{P} \to \operatorname{Aut}\big(\mathcal{S}(\widetilde{P};\mathcal{V})\big).$$

For every $D \in \mathcal{D}^2$ and $\varphi_{P} \in \mathcal{S}(\widetilde{P};\mathcal{V})$, let $D_{\varphi_P} \in \mathcal{D}(N^{\mathbf{s}})$ be the distribution defined by $D_{\varphi_P}(\varphi_{N}) := D(\varphi_{N}\otimes\varphi_{P})$ for all $\varphi_{N} \in \mathcal{S}(N^{\mathbf{s}})$. Then $D_{\varphi_P}\big(\lambda^3_{N}(n)\varphi_{N}\big) = \bar{\psi}^*(n)\,D_{\varphi_P}(\varphi_{N})$ for all $\varphi_{N} \in \mathcal{S}(N^{\mathbf{s}})$ and $n \in N^{\mathbf{s}}$, since $\lambda^2_{N}(n)(\varphi_{N}\otimes\varphi_{P}) = \lambda^3_{N}(n)\varphi_{N}\otimes\varphi_{P}$. By the uniqueness of left quasi-invariant distributions on $N^{\mathbf{s}}$ (cf. §1.18 of [1] – the proof for quasi-invariant distributions is similar), it follows that D_{φ_P} is a constant multiple of the distribution $\bar{\psi}^*\,dn \in \mathcal{D}(N^{\mathbf{s}})$ defined by:

$$\varphi_{N} \mapsto \int_{N^{\mathbf{s}}} \varphi_{N}(n)\,\bar{\psi}^*(n)\,dn, \qquad \forall\, \varphi_{N} \in \mathcal{S}(N^{\mathbf{s}}).$$

Hence, $D_{\varphi_P}(\rho^3_N(n_\circ)\varphi_N) = \psi^*(n_\circ)\, D_{\varphi_P}(\varphi_N)$ for all $\varphi_N \in \mathcal{S}(N^{\mathbf{s}})$ and $n_\circ \in N^{\mathbf{s}} \cap \tilde{w}\widetilde{P}\tilde{w}^{-1}$, and it follows that $\mathcal{D}^2$ is isomorphic to the space $\mathcal{D}^3$ of $\mathcal{V}$-distributions $D \in \mathcal{D}(\widetilde{P};\mathcal{V})$ that satisfy:

$$D\big(\lambda^3_P(p_\circ)\rho^3_P(p)\varphi\big) = \psi^*(\tilde{w}p_\circ\tilde{w}^{-1})\,\delta(p)^{-1/2}\,D\big(\pi_P(p^{-1})\circ\varphi\big)$$

for all $p_\circ \in \tilde{w}^{-1}N^{\mathbf{s}}\tilde{w} \cap \widetilde{P}$, $p \in \widetilde{P}$, and $\varphi \in \mathcal{S}(\widetilde{P};\mathcal{V})$.

Proceeding as above, we next identify $\mathcal{S}(\widetilde{P};\mathcal{V})$ with $\mathcal{S}(\widetilde{M};\mathcal{V}) \otimes \mathcal{S}(U^{\mathbf{s}})$. Thus, for every $\varphi_M \in \mathcal{S}(\widetilde{M};\mathcal{V})$ and $\varphi_U \in \mathcal{S}(U^{\mathbf{s}})$, $\varphi_M \otimes \varphi_U \in \mathcal{S}(\widetilde{P};\mathcal{V})$ is defined by:

$$(\varphi_M \otimes \varphi_U)(mu) := \varphi_M(m)\,\varphi_U(u), \qquad \forall\, m \in \widetilde{M},\ u \in U^{\mathbf{s}}.$$

Let λ^4_M and λ^4_U be the representations defined by left translation:

$$\lambda^4_M : \widetilde{M} \to \mathrm{Aut}\big(\mathcal{S}(\widetilde{M};\mathcal{V})\big), \qquad \lambda^4_U : U^{\mathbf{s}} \to \mathrm{Aut}\big(\mathcal{S}(U^{\mathbf{s}})\big),$$

and let ρ^4_M and ρ^4_U be the representations defined by right translation:

$$\rho^4_M : \widetilde{M} \to \mathrm{Aut}\big(\mathcal{S}(\widetilde{M};\mathcal{V})\big), \qquad \rho^4_U : U^{\mathbf{s}} \to \mathrm{Aut}\big(\mathcal{S}(U^{\mathbf{s}})\big).$$

If $D \in \mathcal{D}^3$ and $\varphi_M \in \mathcal{S}(\widetilde{M};\mathcal{V})$, let $D_{\varphi_M} \in \mathcal{D}(U^{\mathbf{s}})$ be defined by $D_{\varphi_M}(\varphi_U) := D(\varphi_M \otimes \varphi_U)$ for all $\varphi_U \in \mathcal{S}(U^{\mathbf{s}})$. Since $\delta\big|_{U^{\mathbf{s}}} = 1$ and $\pi_P\big|_{U^{\mathbf{s}}} = 1$:

$$D_{\varphi_M}(\rho^4_U(u)\varphi_U) = D(\varphi_M \otimes \rho^4_U(u)\varphi_U) = D\big(\rho^3_P(u)(\varphi_M \otimes \varphi_U)\big) = D(\varphi_M \otimes \varphi_U) = D_{\varphi_M}(\varphi_U)$$

for all $\varphi_U \in \mathcal{S}(U^{\mathbf{s}})$ and $u \in U^{\mathbf{s}}$. Then by the uniqueness of right-invariant distributions on $U^{\mathbf{s}}$ (cf. §1.18 of [1]), it follows that D_{φ_M} is a constant multiple $D'(\varphi_M)$ of the Haar measure $du \in \mathcal{D}(U^{\mathbf{s}})$:

$$D(\varphi_M \otimes \varphi_U) = D'(\varphi_M) \int_{U^{\mathbf{s}}} \varphi_U(u)\,du, \qquad \forall\, \varphi_M \in \mathcal{S}(\widetilde{M};\mathcal{V}),\ \varphi_U \in \mathcal{S}(U^{\mathbf{s}}),$$

and D' is a $\mathcal{V}$-distribution in $\mathcal{D}(\widetilde{M};\mathcal{V})$.

We will now show that if $w \neq w_M$, then $\mathcal{D}^1(N^{\mathbf{s}}\tilde{w}\widetilde{P}) = 0$. Let $D \in \mathcal{D}^3$ be fixed, and let $D' \in \mathcal{D}(\widetilde{M};\mathcal{V})$ be as above. For every $\varphi \in \mathcal{S}(\widetilde{P};\mathcal{V})$, let $\varphi' \in \mathcal{S}(\widetilde{M};\mathcal{V})$ be defined by:

$$\varphi'(m) := \int_{U^{\mathbf{s}}} \varphi(mu)\,du, \qquad \forall\, m \in \widetilde{M}.$$

Then $D'(\varphi') = D(\varphi)$ for all $\varphi \in \mathcal{S}(\widetilde{P};\mathcal{V})$. Indeed, this is easy to check when φ is of the form $\varphi_{_M}\otimes\varphi_{_U}$, and the general case follows by linearity. Since $\widetilde{M}$ normalizes $U^{\mathbf{s}}$, it follows that $\big(\lambda^3_{_P}(u)\varphi\big)' = \varphi'$ for all $\varphi \in \mathcal{S}(\widetilde{P};\mathcal{V})$ and $u \in U^{\mathbf{s}}$. In particular:

$$D(\varphi) = D'(\varphi') = D'\Big(\big(\lambda^3_{_P}(u_\circ)\varphi\big)'\Big) = D\big(\lambda^3_{_P}(u_\circ)\varphi\big) = \psi^*(\tilde{w}u_\circ\tilde{w}^{-1})\,D(\varphi)$$

for all $u_\circ \in \tilde{w}^{-1}N^{\mathbf{s}}\tilde{w}\cap U^{\mathbf{s}}$. Since w is not the longest element in $[W/W_{_M}]$, there exists a positive simple root α such that the root group N_α is contained in wUw^{-1}. Since ψ is *principal*, we have that $\psi^*\big|_{N^{\mathbf{s}}_\alpha} \neq 1$. Moreover, from the definition of the Matsumoto 2-cocycle σ, it follows that $\tilde{w}^{-1}N^{\mathbf{s}}_\alpha\tilde{w} = \mathbf{s}(w^{-1}N_\alpha w)$, and $\tilde{w}^{-1}N^{\mathbf{s}}\tilde{w}\cap U^{\mathbf{s}} = \mathbf{s}(w^{-1}Nw\cap U)$. Hence there exists a $u_\circ \in \tilde{w}^{-1}N^{\mathbf{s}}_\alpha\tilde{w} \subseteq \tilde{w}^{-1}N^{\mathbf{s}}\tilde{w}\cap U^{\mathbf{s}}$ such that $\psi^*(\tilde{w}u_\circ\tilde{w}^{-1}) \neq 1$. This shows that $D = 0$, and therefore $\mathcal{D}^1(N^{\mathbf{s}}\tilde{w}\widetilde{P}) \cong \mathcal{D}^2 \cong \mathcal{D}^3 = 0$.

Now suppose that $w = w_{_M}$. Then $\tilde{w}^{-1}N^{\mathbf{s}}\tilde{w}\cap\widetilde{P} = \tilde{w}^{-1}N^{\mathbf{s}}\tilde{w}\cap\widetilde{M}$, and $\mathcal{D}^3$ is isomorphic to the space $\mathcal{D}^4$ of $\mathcal{V}$-distributions $D \in \mathcal{D}(\widetilde{M};\mathcal{V})$ that satisfy:

$$D\big(\lambda^4_{_M}(m_\circ)\rho^4_{_M}(m)\varphi_{_M}\big) = \psi^*(\tilde{w}m_\circ\tilde{w}^{-1})\,\delta(m)^{1/2}\,D\big(\pi_{_M}(m^{-1})\circ\varphi_{_M}\big)$$

for all $m_\circ \in \tilde{w}^{-1}N^{\mathbf{s}}\tilde{w}\cap\widetilde{M}$, $m \in \widetilde{M}$, and $\varphi_{_M} \in \mathcal{S}(\widetilde{M};\mathcal{V})$. Here we have used the fact that:

$$\rho^3_{_P}(m)(\varphi_{_M}\otimes\varphi_{_U}) = \rho^4_{_M}(m)\varphi_{_M}\otimes\lambda^4_{_U}(m^{-1})\rho^4_{_U}(m)\varphi_{_U}$$

for all $m \in \widetilde{M}$, $\varphi_{_M} \in \mathcal{S}(\widetilde{M};\mathcal{V})$, and $\varphi_{_U} \in \mathcal{S}(U^{\mathbf{s}})$, which implies that:

$$\big(\rho^3_{_P}(m)(\varphi_{_M}\otimes\varphi_{_U})\big)' = \delta(m)\big(\rho^4_{_M}(m)\varphi_{_M}\otimes\varphi_{_U}\big)'.$$

By another straightforward calculation with the 2-cocycle σ, $\tilde{w}^{-1}N^{\mathbf{s}}\tilde{w}\cap\widetilde{M} = N^{\mathbf{s}}_{_M}$. Thus, from the definition of $\bar{\psi}^*_{_M}$, it follows that complex conjugation provides an isomorphism between $\mathcal{D}^4$ and the space $\mathcal{D}^5$ of $\mathcal{V}$-distributions $D \in \mathcal{D}(\widetilde{M};\mathcal{V})$ that satisfy:

$$D\big(\lambda^4_{_M}(m_\circ)\rho^4_{_M}(m)\varphi\big) = \bar{\psi}^*_{_M}(n_\circ)\,\delta(m)^{-1/2}\,D\big((\delta^{-1}\otimes\pi_{_M})(m^{-1})\circ\varphi\big), \qquad \forall\; n_\circ \in N^{\mathbf{s}}_{_M},\; m \in \widetilde{M},$$

for all $\varphi \in \mathcal{S}(\widetilde{M};\mathcal{V})$. By Bruhat's theorem, $\mathcal{D}^5 \cong \mathrm{Bil}_{\widetilde{M}}({}^\circ\Pi^{\bar{\psi}}_{_M}, \delta^{-1}\otimes\pi_{_M})$, the space of intertwining forms of ${}^\circ\Pi^{\bar{\psi}}_{_M}$ and $\delta^{-1}\otimes\pi_{_M}$, which is isomorphic to $\mathrm{Hom}_{\widetilde{M}}(\delta^{-1}\otimes\pi_{_M}, \Pi^{\psi}_{_M})$. Finally, since $\delta\big|_{N^{\mathbf{s}}_{_M}} = 1$, the spaces $\mathrm{Hom}_{\widetilde{M}}(\delta^{-1}\otimes\pi_{_M}, \Pi^{\psi}_{_M})$ and $\mathrm{Hom}_{\widetilde{M}}(\pi_{_M}, \Pi^{\psi}_{_M})$ are isomorphic (although the representations $\delta^{-1}\otimes\pi_{_M}$ and $\pi_{_M}$ need not be). Thus, in the case $w = w_{_M}$, we have that $\mathcal{D}^1(N^{\mathbf{s}}\tilde{w}\widetilde{P}) \cong \mathrm{Hom}_{\widetilde{M}}(\pi_{_M}, \Pi^{\psi}_{_M})$.

As $\widetilde{G}$ is a finite covering of G, we have that $N^{\mathbf{s}}\tilde{w}_{_M}\widetilde{P}$ is open in $\widetilde{G}$, since $Nw_{_M}P$ is open in G. Similarly, $X^1 := \coprod_{w\neq w_M} N^{\mathbf{s}}\tilde{w}\widetilde{P}$ is closed in $\widetilde{G}$, and the sequence:

$$(**) \qquad 0 \to \mathcal{D}^1(X^1) \to \mathcal{D}^1(\widetilde{G}) \to \mathcal{D}^1(N^{\mathbf{s}}\tilde{w}_{_M}\widetilde{P}) \to 0$$

is exact (cf. §1.9 of [1]). We will now show that $\mathcal{D}^1(X^1) = 0$. Indeed, for $i \geq 1$, let $X^{i+1} := X^i - Y^i$, where:

$$Y^i := \coprod_{w\in W^i} N^{\mathbf{s}}\tilde{w}\widetilde{P}, \qquad W^i := \left\{w \in W \;\middle|\; N^{\mathbf{s}}\tilde{w}\widetilde{P} \text{ is an open subset of } X^i\right\}.$$

Then $W - \{w_{_M}\} = \coprod_{i\geq 1} W^i$, and $X^1 = \coprod_{i\geq 1} Y^i$. As each Y^i is open in X^i, and each X^{i+1} is closed in X^i, the sequence:

$$0 \to \mathcal{D}^1(X^{i+1}) \to \mathcal{D}^1(X^i) \to \mathcal{D}^1(Y^i) \to 0$$

is also exact. But $N^{\mathbf{s}}\tilde{w}\widetilde{P}$ is also an open subset of Y^i for every $w \in W^i$, hence it follows that $\mathcal{D}^1(Y^i) \cong \bigoplus_{w\in W^i} \mathcal{D}^1(N^{\mathbf{s}}\tilde{w}\widetilde{P}) = 0$. Consequently, $\mathcal{D}^1(X^1) \cong \mathcal{D}^1(X^i)$ for all $i \geq 1$, and since $X^i = \emptyset$ for $i \gg 0$, this shows that $\mathcal{D}^1(X^1) = 0$. From the exact sequence $(**)$, it now follows that $\mathcal{D}^1(\widetilde{G}) \cong \mathcal{D}^1(N^{\mathbf{s}}\tilde{w}_{_M}\widetilde{P}) \cong \mathrm{Hom}_{\widetilde{M}}(\pi_{_M}, \Pi^{\psi}_{_M})$, and this completes the proof. □

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