Math 4100 typical problems for Exam II

1. Suppose that
$$f(t) = \begin{cases} t, & 0 \le t < 4, \\ 3, & t \ge 4. \end{cases}$$
Determine the steady state response for the system
$$y'' + y' + 4y = f(t).$$
Answer: $y_{ss}(t) = 3/4$.

2. Use Laplace transforms to solve the IVP
$$\ddot{u} + u = 1 + \sin 4t, \qquad u(0) = 1, \quad u'(0) = 0.$$
Answer: $u(t) = 1/15(15 + 4\sin t - \sin 4t)$

3. Find the general solution of
$$x'' + \frac{1}{2}x' + x = 0.$$
Answer: $x(t) = e^{-t/4}(\alpha \cos \sqrt{15}/4t + \beta \sin \sqrt{15}/4t$

4. Find the general solution of
$$t^2\ddot{u} - t\dot{u} + u = t^2.$$
Hint: To find the homogeneous solution try t^r.
Answer: $u(t) = t^2 + \alpha t + \beta t \ln t$

5. Find the amplitude, period, and phase shift for the steady state response of the periodically forced oscillator
$$y'' + y' + 2y = 3\cos t.$$
Answer: $y_{ss}(t) = 3/2(\cos t + \sin t)$. The amplitude is $3/2\sqrt{2}$, the phase shift is $\pi/4$, and the period is 2π.

Figure 1: Graph of $y(t) = e^{-2t}(1 - 2e^t)$

6. Solve the differential equation

$$\ddot{u} + u = u_3(t)\sin(t/2), \qquad u(0) = 0, \quad u'(0) = 0.$$

Hint: Make sure you have the correct translations before you apply the Laplace transform.
Answer:
$$u_3(t)[\frac{4}{3}\sin\frac{3}{2}(\cos(\frac{1}{2}(t-3)) - \cos(t-3)) + \frac{2}{3}\cos\frac{3}{2}(2\sin(\frac{1}{2}(t-3)) - \sin(t-3))]$$

7. Use the definition of the Laplace transform to find the Laplace transforms of e^{at}, t, and $\cos at$.

8. Find the general solution of

$$y'' + y = 3 - \cos t.$$

Answer: $y(t) = 3 - 1/2t\sin t + \alpha\cos t + \beta\sin t$

9. Suppose that a certain mass-spring system in modeled by

$$y'' + 3y' + 2y = 0,$$

where y measures the displacement of the mass from equilibrium. Graph the motion of the mass if the spring is compressed one unit and then released from rest.

Answer: $y(t) = e^{-2t}(1 - 2e^t)$

10. Use Laplace transforms to solve the IVP

$$\ddot{u} + u = \sin 4t\delta(t - 2), \qquad u(0) = 0, \quad u'(0) = 0.$$

Remember that $\int_{-\infty}^{\infty} f(t)\delta(t - t_0)\,dt = f(t_0)$.
Answer: $u(t) = \sin(8)u_2(t)\sin(t - 2)$