Math 4100 Exam III Study Sheet

Note: The answers provided here were computed in Mathematica. You can do the same thing and find the answer to almost every problem in this course by using an algebraic processor such as Mathematica or Maple.

1. Solve the IVP for the system of first-order ODEs

$$\begin{aligned}\dot{x} &= -2x + 3y,\\ \dot{y} &= 3x - 2y\end{aligned}$$

with $x(0) = 1$ and $y(0) = 2$. Answer:

$$x(t) = \frac{1}{2}e^{-5\,t} + \frac{3}{2}e^{t}, \qquad y(t) = -\frac{1}{2}e^{-5\,t} + \frac{3}{2}e^{t}.$$

2. Find the general solution of the system of first-order ODEs

$$\begin{aligned}\dot{x} &= 3x + 2y,\\ \dot{y} &= -2x + 3y.\end{aligned}$$

Answer:

$$x(t) = e^{3\,t}C_1\,\cos(2\,t) + e^{3\,t}\,C_2\,\sin(2\,t), \qquad y(t) = e^{3\,t}\,C_2\,\cos(2\,t) - e^{3\,t}\,C_1\,\sin(2\,t).$$

3. Solve the IVP

$$\begin{aligned}\dot{x} &= 2x + \frac{3}{2}y,\\ \dot{y} &= -\frac{3}{2}x - y,\\ x(0) &= 3,\\ y(0) &= -2\end{aligned}$$

Answer:

$$x(t) = \frac{3\,e^{\frac{t}{2}}\,(2+t)}{2}, \qquad y(t) = \frac{-e^{\frac{t}{2}}\,(4+3\,t)}{2}.$$

4. Find the general solution of the system of first-order ODEs

$$\begin{aligned}\dot{x} &= 3x + y,\\ \dot{y} &= -x + y.\end{aligned}$$

Answer:

$$x(t) = C_2\,e^{2\,t}\,t + C_1\,e^{2\,t}\,(1+t)\,, \qquad y(t) = -C_1e^{2\,t}\,t - e^{2\,t}\,(-1+t)\,C_2.$$

5. Find the general solution of the system of first-order ODEs

$$\begin{aligned} \dot{x} &= 2x - 5y - \cos t, \\ \dot{y} &= x - 2y + \sin t. \end{aligned}$$

Answer:

$$\begin{aligned} x(t) &= -(\frac{1}{2} + 2\,t)\,\cos(t) - 5\,C_2\,\sin(t) + (1-t)\,\sin(t) + C_1\,(\cos(t) + 2\,\sin(t))\,, \\ y(t) &= (-1+t)\,\cos(t) + \frac{\sin(t)}{2} + C_2\,(\cos(t) - 2\,\sin(t)) + C_1\,\sin(t). \end{aligned}$$

6. Consider the IVP $\dot{y} = y^2 + t, \quad y(0) = 0$. (a) Find the Euler approximation for $y(1)$ for the step size $h = 1/2$. (b) Find the improved Euler approximation for $y(1)$ for the step size $h = 1/2$. (c) Suppose the error for the improved Euler approximation is $1/5$. What would you expect the error to be if $h = 1/8$. Answer: (a) $(t_n, y_n) = \{\{0,0\},\{\frac{1}{2},0\},\{1,\frac{1}{4}\}\}$, for $n = 0,1,2$. (b) $\{\{0,0\},\{\frac{1}{2},\frac{1}{8}\},\{1,\frac{35425}{65536}\}\}$.