MATH 4560 Take Home Final Exam
Professor Chicone Fall 2004
Due Friday 9 December 3:00 P.M.

Note: This exam has a mix of pencil and paper and computer experiments. I hope you will not be overwhelmed with programming in Mathematica! Let me also mention that I tried to be careful not to make a mistake. But, if you think a problem is incorrectly stated, you should let me know or you can explain what you think is wrong with the problem. Finding errors is good! It shows that you understand what you are doing. I hope you find some interesting problems and pictures. Good luck.

1. Suppose that a particle moves with the flow of the first-order system $\dot{x} = x(4 - x - y)$, $\dot{y} = y(x - 2)$ starting at the point $(x, y) = (1/2, 2)$ at time $t = 0$. Where will the particle be after a long time has passed? Explain.

2. (a) Draw the phase portrait of the system

$$\dot{x} = y, \qquad \dot{y} = x - x^2.$$

(b) Give a pencil and paper argument to show that the system has periodic orbits. (c) Is the solution with initial condition $(x, y) = (3/2, 0)$ periodic? Explain.

3. Write a first-order system that is equivalent to the differential equation

$$\ddot{x} + \dot{x}(x^2 + \dot{x}^2 - 1) + x = 0.$$

Show that the system has a limit cycle. What is the stability type of the limit cycle?

4. (a) Determine the Hopf bifurcation curve in the (a, b)-parameter space for the system

$$\dot{x} = ax - y - x^3, \qquad \dot{y} = x + by - y^3.$$

(b) Based on pencil and paper computations, determine the set of parameter values where a stable limit cycle exists. Verify your answer to parts (a) and (b) with computer experiments.

5. Consider the one-dimensional map $f(x) = r\sin(\pi x)$ as a dynamical system on the unit interval, where the parameter r satisfies $0 < r \leq 1$. Use computer experiments to show that this map goes through the period doubling route to chaos. Also, use computer experiments to compute an estimate of the Feigenbaum number (see Exercises 10.6.1 and 10.6.2).

6. (a) Recreate Figure 12.5.6. The system is

$$\ddot{x} + \lambda\dot{x} - x + x^3 = F\cos\omega t.$$

The figure shows the attractor for the Poincaré map corresponding to the parameter values $\lambda = 0.25$, $\omega = 1$ and $F = 0.40$.

7. The polynomial equation $z^3 - 1 = 0$ has three roots in the complex plane. Find these roots. Write a program to approximate these known roots using Newton's method ($x_{n+1} = x_n - f(x_n)/f'(x_n)$). In Mathematica you can use complex arithmetic: If you put in a complex starting value in the form `x+Iy`, the function values will be complex. Real and imaginary parts can be extracted with `Re[z]` and `Im[z]`. Consider a square box in the complex plane centered at the origin with one corner at $2 + 2i$. You can adjust the size of this box later. The box should contain all three roots! Next consider a small disc around each of the roots. Name the roots "red", "green", and "blue". For debugging purposes you could make the disc have radius 0.1, but you may wish to consider smaller radii later. Now, here is the experiment. Choose a grid of points covering the box (random choices of points is another possibility). For each point, save the starting value and start iterating Newton's method for finding the roots of $z^3 - 1 = 0$. Stop the iteration as soon as an iterate falls into one of the disks, an iterate has modulus larger than 10, or at least 500 iterates have been taken. (A smaller disc size and a larger value of the maximum number of iterates will result in a better picture.) If the iterate lands in one of the disks, then color the starting point according to the color of the corresponding root (red, green, or blue). If the modulus is too large discard the point (i.e. don't color it) and if the maximum number of iterates are taken, then color the point black. Do the experiment for lots of points (hundreds for debugging; thousands or more for the final runs) and plot the colored points in the box. Pay attention to the boundaries between the red, green, and blue "regions". What do you see? You should blow up a small box containing part of the boundary to see more detail. (Of course, this will require more computation!) Describe what you see.