MATH 4560 Take Home Final Exam
Professor Chicone Fall 2007
Due (on or before) Monday 10 December 12:00 Noon.

I tried to be careful not to make a mistake on this exam. But, if you think a problem is incorrectly stated, you should let me know or you can explain in your report what you think is wrong with the problem. Finding errors is good! It shows that you understand what you are doing. I hope you enjoy doing some of the problems on this exam. Good luck.

1. Suppose that a particle moves with the flow of the first-order system $\dot{x} = x(4 - x - y)$, $\dot{y} = y(x - 2)$ starting at the point $(x, y) = (1/2, 2)$ at time $t = 0$. Where will the particle be after a long time has passed? Explain using pencil and paper.

2. (a) Draw the phase portrait of the system

$$\dot{x} = y, \qquad \dot{y} = x - x^2.$$

   (b) Give a pencil and paper argument to show that the system has periodic orbits. (c) Is the solution with initial condition $(x, y) = (3/2, 0)$ periodic? Explain.

3. Write a first-order system that is equivalent to the differential equation

$$\ddot{x} + \dot{x}(x^2 + \dot{x}^2 - 1) + x = 0.$$

   Show that the system has a limit cycle. What is the stability type of the limit cycle?

4. (a) Determine the Hopf bifurcation curve in the (a, b)-parameter space for the system

$$\dot{x} = ax - y - x^3, \qquad \dot{y} = x + by - y^3.$$

   (b) Based on pencil and paper computations, determine which "side" of the bifurcation curve (if any) corresponds to the existence of stable limit cycles. Verify your answer to parts (a) and (b) with computer experiments. (c) Give a pencil and paper argument to prove that the system for $a = b = 1$ has a stable limit cycle.

5. Consider the one-dimensional map $f(x) = r\sin(\pi x)$ as a dynamical system on the unit interval, where the parameter r satisfies $0 < r \leq 1$. Use computer experiments to show that this map goes through the period doubling route to chaos. Also, use computer experiments to compute an estimate of the Feigenbaum number (see Exercises 10.6.1 and 10.6.2).

6. (a) Recreate Figure 12.5.6. The system is

$$\ddot{x} + \lambda\dot{x} - x + x^3 = F\cos\omega t.$$

   The figure shows the attractor for the Poincaré map corresponding to the parameter values $\lambda = 0.25$, $\omega = 1$ and $F = 0.40$. (b) Set $\lambda = 0.25$ and $\omega = 1$ and consider F as a bifurcation parameter. Explore the dynamics of the system with F over the range $0 \leq F \leq 1$. Pick no more than six parameter values that correspond to different dynamics, make pictures and *explain in words what each picture is supposed to show.*

7. Do problem 12.1.9 on the dynamical system obtained by iteration of the standard map

$$P(x, y) = (x + y + k\sin x, y + k\sin x).$$

   It is very important to make sure that your code evaluates the variables modulo 2π. Your computer window should be $[0, 2\pi] \times [0, 2\pi]$ or $[-\pi, \pi] \times [-\pi, \pi]$. Iterates that fall outside the window are moved back into the window by adding or subtracting an appropriate multiple of 2π.

8. **Extra Credit Problem:** Choose a dynamical system that has come up in one of your other classes, in your research project, or from some other source and analyze it using the methods you have learned in this class. Report on your findings.