Asymptotic Phase Revisited

Carmen Chicone[*] and Weishi Liu[†]

December 23, 2003

1 Introduction

We will consider asymptotic phase for periodic orbits of smooth planar vector fields (cf. [4, 5, 9, 10, 12]). Our main result is that, for most nonhyperbolic limit cycles, the only points in phase with points on the limit cycle are points on the limit cycle itself. More precisely, suppose that Γ is a nonhyperbolic limit cycle of a C^2 planar system, Σ is a transverse section at $p \in \Gamma$, τ is the time of first return to Σ, and P is the corresponding Poincaré map. Since Γ is not hyperbolic, $P'(p) = 1$. Under the assumption that Γ is a generic nonhyperbolic limit cycle (that is, $P''(p) \neq 0$), we will show that $\tau'(p) = 0$ is a necessary and sufficient condition for every point in some neighborhood of Γ to be in phase with a point on Γ. The C^2 requirement on the corresponding planar system is crucial. In fact, for each $\alpha \in (0, 1)$, we will give an example (Example 1 in Section 3) of a $C^{1+\alpha}$ planar system with a nonhyperbolic limit cycle Γ such that $\tau'(p) \neq 0$ but every point in some neighborhood of Γ is in phase with a point on Γ. We remark that, if $m \geq 2$ and the Poincaré map P is given by

$$P(\sigma) = \sigma + c\sigma^m + o(\sigma^m) \quad \text{for some} \quad c \neq 0$$

then Γ is isochronous if and only if

$$\tau'(p) = \tau''(p) = \cdots = \tau^{m-1}(p) = 0.$$

---

[*] Department of Mathematics, University of Missouri, Columbia, MO 65211.
E-mail: carmen@chicone.math.missouri.edu

[†] Department of Mathematics, University of Kansas, Lawrence, KS 66045.
E-mail: wliu@math.ukans.edu. Partially supported by NSF Grant DMS-0071931.

To complete the picture for the nongeneric case, we will construct two C^∞ planar systems each with a nonhyperbolic limit cycle Γ such that $\tau^{(n)}(p) = 0$ for all $n \geq 1$ and $P^{(n)}(p) = 0$ for all $n \geq 2$. In one of these systems Γ is nonisochronous (Example 2); in the other, Γ is isochronous (Example 3).

For the case of hyperbolic periodic orbits, the existence of invariant foliations and asymptotic phase is well known. These results have also be generalized to normally hyperbolic invariant manifolds [6, 7, 8, 11] (see also [2, 3]). We will give a new proof to show that hyperbolic limit cycles have invariant foliations (cf. [9]); hence, every point in some neighborhood of every hyperbolic limit cycle has asymptotic phase with respect to the limit cycle.

Finally, we mention that some of our results can be generalized to periodic orbits of smooth vector fields in $\mathbb{R}^n$.

2 Isochronous Planar Limit Cycles

Let Γ be a periodic orbit of a flow ϕ_t defined on $\mathbb{R}^n$. A point $q \in \mathbb{R}^n$ has *asymptotic phase* with respect to Γ if there is a point $p \in \Gamma$ such that $\lim_{t\to\infty} |\phi_t(q) - \phi_t(p)| = 0$ or $\lim_{t\to-\infty} |\phi_t(q) - \phi_t(p)| = 0$. In this case, q is also said to be *in phase* with p. We will call a limit cycle Γ *isochronous* if there is an (open) neighborhood containing Γ such that every point in the neighborhood is in phase with a point on Γ.

In this section, Γ is a periodic orbit of the ODE

$$\dot{u} = f(u) \tag{1}$$

Σ' is a transverse section at $p \in \Gamma$, $\tau : \Sigma \to \mathbb{R}$ is the return-time function defined on a subsection Σ that contains p, and $T := \tau(p)$ is the period of Γ. Rather than repeat this context each time it is needed, we will often refer to the return-time function or its derivatives "at Γ" when we mean "at a point $p \in \Gamma$ along a transverse section Σ". This language will only be used when the corresponding statement about T does not depend on the choice of the section or the point on Γ.

We begin with the linear structure of the flow at Γ. The main result here is the existence of a linear space N_p at p that is ϕ_T-invariant and transverse to the one-dimensional vector space spanned by $f(p)$. In the hyperbolic case, the proof of the existence of N_p follows from the structure of the linear map $D\phi_T(p)$ by linear algebra. All that is needed is the observation that $f(p)$ is the generator of a one-dimensional eigenspace corresponding to the eigenvalue one. In the planar case, we will obtain an explicit formula for the generator (if it exists) of the one-dimensional invariant complementary space.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(x,y) = (f_1(x,y), f_2(x,y))$ be smooth. Define $f^\perp = Rf$, where R is the rotation matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$; the Euclidean divergence and curl

$$\begin{aligned} \operatorname{div} f(x,y) &:= \frac{\partial f_1}{\partial x}(x,y) + \frac{\partial f_2}{\partial y}(x,y), \\ \operatorname{curl} f(x,y) &:= \frac{\partial f_2}{\partial x}(x,y) - \frac{\partial f_1}{\partial y}(x,y); \end{aligned}$$

and the scalar curvature function along the smooth curve $t \mapsto (x(t), y(t))$ by

$$\kappa := \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

Theorem 2.1 (Diliberto's Theorem). *Let ϕ_t denote the flow of the differential equation (1). If $f(\zeta) \neq 0$, then the principal fundamental matrix solution $t \mapsto D\phi_t(\zeta)$ at $t = 0$ of the homogeneous variational equation*

$$\dot{W} = Df(\phi_t(\zeta))W$$

is such that

$$D\phi_t(\zeta)f(\zeta) = f(\phi_t(\zeta)), \quad D\phi_t(\zeta)f^\perp(\zeta) = a(t,\zeta)f(\phi_t(\zeta)) + b(t,\zeta)f^\perp(\phi_t(\zeta)),$$

where

$$\begin{aligned} b(t,\zeta) &= \frac{|f(\zeta)|^2}{|f(\phi_t(\zeta))|^2}\, e^{\int_0^t \operatorname{div} f(\phi_s(\zeta))\, ds}, && (2) \\ a(t,\zeta) &= \int_0^t \big(2\kappa(s,\zeta)|f(\phi_s(\zeta))| - \operatorname{curl} f(\phi_s(\zeta))\big) b(s,\zeta)\, ds. && (3) \end{aligned}$$

Proposition 2.2. *The functions a and b satisfy the identities*

$$a(s+t,\zeta) = a(s,\zeta) + b(s,\zeta)a(t,\phi_s(\zeta)), \qquad b(s+t,\zeta) = b(s,\zeta)b(t,\phi_s(\zeta)) \quad (4)$$

for all $t \in \mathbb{R}$ and $\zeta \in \Gamma$. The periodic orbit Γ is hyperbolic if and only if $b(T,p) \neq 1$. In case Γ is not hyperbolic, the derivative of τ at p is a nonzero scalar multiple of $a(T,p)$. In addition, suppose that $h(p)$ is the positive direction vector which is tangent to Σ at p with respect to the local coordinate on Σ. If the ordered set of vectors $\{f(p), h(p)\}$ is positively oriented with respect to the usual orientation of the plane, then the derivative of τ at p is a positive scalar multiple of $a(T,p)$.

The proofs of Diliberto's theorem and the latter three statements of Proposition 2.2 are in [5]. The identities (4) are proved by applying appropriate changes of variables in the integrals that appear in the definitions of a and b.

Using (4), it is easy to prove the additional identities

$$\begin{aligned} a(t+T,\zeta) &= a(T,\zeta)+b(T,\zeta)a(t,\zeta), && (5)\\ b(t+T,\zeta) &= b(t,\zeta)b(T,\zeta), && (6)\\ b(t,\zeta)a(T,\phi_t(\zeta)) &= a(T,\zeta)+(b(T,\zeta)-1)a(t,\zeta), && (7)\\ b(T,\phi_t(\zeta)) &= b(T,\zeta). && (8) \end{aligned}$$

Proposition 2.3 (Invariant Normal Bundle). *Suppose that Γ is a periodic orbit of the ODE (1). If Γ is hyperbolic, then Γ has a unique invariant normal bundle generated by the vector field given by*

$$g(\zeta)=\frac{a(T,\zeta)}{(b(T,\zeta)-1)}f(\zeta)+f^{\perp}(\zeta). \tag{9}$$

In case Γ is not hyperbolic, it has an invariant normal bundle if and only if the derivative of the return-time map vanishes on Γ; that is, $\tau'(p)=0$. In this case there are infinitely many distinct normal bundles each generated by

$$g(\zeta)=A(\zeta)f(\zeta)+f^{\perp}(\zeta), \tag{10}$$

where $p\in\Gamma$, c is a real number, and (for $\zeta=\phi_t(p)$) $A(\zeta):=(c+a(t,p))/b(t,p)$. In both cases, $D\phi_T(p)g(p)=b(T,p)g(p)$.

Proof. It suffices to find functions A and B defined on Γ so that B does not vanish and the vector field g defined on Γ by $g(\zeta)=A(\zeta)f(\zeta)+B(\zeta)f^{\perp}(\zeta)$ is such that $D\phi_t(\zeta)g(\zeta)=\lambda(t,\zeta)g(\phi_t(\zeta))$ for some nonvanishing scalar function $\lambda:\mathbb{R}\times\Gamma\to\mathbb{R}$. Using Diliberto's theorem, appropriate A and B exist if and only if, for all $\zeta\in\Gamma$, they satisfy the equations

$$A(\zeta)+B(\zeta)a(t,\zeta)=\lambda(t,\zeta)A(\phi_t(\zeta)),\quad B(\zeta)b(t,\zeta)=\lambda(t,\zeta)B(\phi_t(\zeta)), \tag{11}$$

the nondegeneracy condition $B(\zeta)\neq 0$, and the compatibility condition

$$(b(T,\zeta)-1)A(\zeta)=B(\zeta)a(T,\zeta)$$

(because $\phi_T(\zeta)=\zeta$).

In case Γ is hyperbolic, equivalently $b(T, p) \neq 1$ for some $p \in \Gamma$, we define $B(\zeta) := 1$, $\lambda(t, \zeta) := b(t, \zeta)$, and $A(\zeta) := a(T, \zeta)/(b(T, \zeta) - 1)$. The compatibility condition is satisfied; and, using the identities (7) and (8), we find that A, B and λ satisfy the system of equations (11), as required. Also, we note that the compatibility condition requires that $A(\zeta) = B(\zeta)a(T, \zeta)/(b(T, \zeta) - 1)$. While we take $B(\zeta) = 1$ to obtain equation (9), other choices of B only serve to change the lengths of the vectors $g(\zeta)$, not their directions. Hence, the invariant normal bundle is unique.

If Γ is not hyperbolic, then $b(T, \zeta) \equiv 1$. To satisfy the nondegeneracy and compatibility conditions, we must have $a(T, \zeta) \equiv 0$. Conversely, suppose that $a(T, \zeta) \equiv 0$. Fix $p \in \Gamma$ and let c be a real number. Take $B(p) = 1$, $A(p) = c$, and define $A(\phi_t(p)) := (c + a(t, p))/b(t, p)$. The compatibility condition is satisfied. The first equation of display (11) is equivalent to

$$\frac{c + a(s, p)}{b(s, p)} + a(t, \phi_s(p)) = b(t, \phi_t(p))\frac{c + a(s + t, p)}{b(s + t, p)}.$$

This identity is verified by substituting for $a(t, \phi_s(p))$ and $b(t, \phi_t(p))$ using the identities (4). ☐

Proposition 2.4. *If the transverse section* Σ *at* p *is contained in the section* Σ' *at* p *and* $\phi_T(\Sigma) \subseteq \Sigma'$, *then* Γ *has a smooth invariant normal foliation whose leaves are given by* $\phi_t(\Sigma)$ *for* $t \in [0, T)$. *Moreover, if* Γ *has an invariant normal foliation and* Γ *is a limit cycle, then* Γ *is isochronous. In particular, every point on each leaf of the foliation is in phase with its base point (that is, the intersection of the leaf with* Γ*).*

Proof. The proof is an application of the semigroup property of the flow ϕ_t. ☐

Theorem 2.5. *If* Γ *is hyperbolic, then* Γ *has a smooth* ϕ_T*-invariant normal foliation whose leaves are tangent to the invariant normal bundle generated by* g *in equation (9). In particular,* Γ *is isochronous.*

Proof. By Proposition 2.4 it suffices to show that there is a section Σ' at $p \in \Gamma$ and a subsection Σ at p such that $\phi_T(\Sigma) \subseteq \Sigma'$. Also, by reversing the direction of time, there is no loss of generality if we assume that Γ is attracting, that is $b(T, p) < 1$. Also, without loss of generality, we will assume that f is defined on $\mathbb{R}^2$ and $\dot{u} = f(u)$ is complete.

We will prove the foliation is class C^1; the proof for class C^r is similar. Let $C^1([-1,1],\mathbb{R})$ denote the usual Banach space of continuously differentiable functions with the usual (sum) C^1-norm and define $\mathcal{E} := \{h \in C^1([-1,1],\mathbb{R}) : h(0) = 0 \text{ and } h'(0) = 0\}$. The set $\mathcal{E}$ is a Banach space with respect to the norm $\|h\|_{\mathcal{E}} = \sup_{s\neq 0} |h'(s)|$. To prove this fact, note first that $\|\|_{\mathcal{E}}$ is a norm. In particular, if $\|h\|_{\mathcal{E}} = 0$ then $h'(s) \equiv 0$; therefore h is constant. The function h is zero because $h(0) = 0$. Secondly, suppose that $\{h_n\}_{n=1}^{\infty}$ is $\mathcal{E}$-Cauchy and use the mean value theorem to obtain the estimate

$$|h_m(s)-h_n(s)|+|h'_m(s)-h'_n(s)| \leq |s| \sup_{t\in[-1,1]} |h'_m(t)-h'_n(t)|+|h'_m(s)-h'_n(s)|.$$

It follows that $\|h_m - h_n\|_1 \leq 2\|h_m - h_n\|_{\mathcal{E}}$. Hence, $\{h_n\}_{n=1}^{\infty}$ is C^1-Cauchy. Let h denote its limit in $C^1([-1,1],\mathbb{R})$. Since $|h(0)| + |h'(0)| \leq \|h - h_n\|_1$, we have that $h(0) = h'(0) = 0$, as required.

For $h \in \mathcal{E}$ and $\delta > 0$, we have a curve L in the plane given by $s \mapsto p + \delta h(s) f(p) + \delta s g(p)$, where g is defined in display (9) and $s \in [-1,1]$. This curve is transverse to Γ at p. Also, let $t \mapsto u(t,s,h,\delta)$ denote the solution of $\dot{u} = f(u)$ with the initial condition $u(0,s,h,\delta) = p + \delta h(s) f(p) + \delta s g(p)$. Using the linear independence of $f(p)$ and $g(p)$, the vector $u(T,s,h,\delta)$ can be uniquely expressed in the form $u(T,s,h,\delta) = p + \tilde{y}(T,s,h,\delta) f(p) + \tilde{x}(T,s,h,\delta) g(p)$ for scalar valued functions $\tilde{x}$ and $\tilde{y}$. Thus, the section given by L is ϕ_T-invariant if and only if

$$\delta h(\frac{1}{\delta}\tilde{x}(T,s,h,\delta)) = \tilde{y}(T,s,h,\delta) \tag{12}$$

whenever $s \in [-1,1]$.

To take advantage of the invariant normal bundle defined in Proposition 2.3, let us linearize at Γ. By Taylor's theorem

$$f(\zeta + \eta) = f(\zeta) + Df(\zeta)\eta + R(\zeta,\eta)\eta,$$

where $R(\zeta,\eta) := \int_0^1 (Df(\zeta + s\eta) - Df(\zeta))\, ds$. This formula is valid for $\zeta \in \Gamma$ and η in some open neighborhood of Γ. Moreover, $R(\zeta, 0) \equiv 0$. Using this fact and the definition $w(t,s,h,\delta) := (u(t,s,h,\delta) - \phi_t(p))/\delta$ we have that

$$\dot{w} = Df(\phi_t(p))w + R(\phi_t(p), \delta w)w \tag{13}$$

with the initial condition $w(0,s,h,\delta) = h(s)f(p) + sg(p)$. Also, with $x := \tilde{x}/\delta$ and $y := \tilde{y}/\delta$, we have the equality

$$w(T,s,h,\delta) = y(T,s,h,\delta)f(p) + x(T,s,h,\delta)g(p).$$

Thus, the curve L is invariant if and only if $h(x(T,s,h,\delta)) = y(T,s,h,\delta)$.

Using the independence of $f(p)$ and $g(p)$, let us define a new norm on $\mathbb{R}^2$ given by $|\zeta|_{\text{fg}} := \max\{|\zeta_f|, |\zeta_g|\}$, where $\zeta = \zeta_f f(p) + \zeta_g g(p)$.

By the usual existence theory for ODEs, w is as smooth as the function f. Let $\mathcal{B}$ denote the ball of radius $1/2$ centered at the origin in $\mathcal{E}$. There is a number $M > 0$ such that $|w(t,s,h,\delta)|_{\text{fg}} \le M$ whenever $t \in [0,T]$, $s \in [-1,1]$, $h \in \mathcal{B}$, and $\delta \in [-1,1]$. This follows because the time interval is compact and the initial data is contained in a compact set. For this result, we note that $|h(s)| \le \|h\|_{\mathcal{E}}|s| \le 1/2$. Thus, the initial data lies in a compact neighborhood of the origin in $\mathbb{R}^2$. Using the usual semilinear Gronwall estimate applied to the ODE (13) (that is, by using the variation of constants formula, making a fg-norm estimate, and then applying Gronwall's lemma), we obtain the inequality

$$\begin{aligned} |w(T,s,h,\delta)|_{\text{fg}} &\le e^{\int_0^T \Upsilon(T,s,\sigma,h,\delta)\,d\sigma} |D\phi_T(p)(h(s)f(p) + sg(p))|_{\text{fg}} \\ &\le e^{\int_0^T \Upsilon(T,s,\sigma,h,\delta)\,d\sigma} \max\{|h(s)|, |s|b(T,p)\} \qquad (14) \end{aligned}$$

where $\Upsilon(T,s,\sigma,h,\delta) := \|D\phi_T(p)(D\phi_\sigma(p))^{-1}R(\phi_\sigma(p), \delta w(\sigma,s,h,\delta))\|_{\text{fg}}$ and the matrix norm is associated with the fg-norm. By restricting δ to a sufficiently small open interval $J \subset [-1,1]$ containing the origin, Υ can be made sufficiently small so that

$$E := \int_0^T \Upsilon(T,s,\sigma,h,\delta)\,d\sigma < \min\{2, \frac{1}{b(T,p)}\}.$$

Thus, using the estimate (14), we have

$$\max\{|x(T,s,h,\delta)|, |y(T,s,h,\delta)|\} \le E \max\{|h(s)|, |s|b(T,p)\}.$$

With $\delta \in J$ and $h \in \mathcal{B}$, it follows that $|x(T,s,h,\delta)| < 1$; in particular, $x(T,s,h,\delta)$ is in the domain of $h \in \mathcal{E}$. Therefore, there is a function $F : \mathcal{B} \times J \to \mathcal{E}$ given by

$$F(h,\delta)(s) = y(T,s,h,\delta) - h(x(T,s,h,\delta)).$$

Since ODEs depend smoothly on their parameters, even for parameters in Banach spaces (see [1, 5]), the functions given by $Y(h,\delta)(s) := y(T,s,h,\delta)$ and $X(h,\delta)(s) := x(T,s,h,\delta)$ are as smooth as the function f. Also, the operator (composition on the left) given by $\Phi(h)(s) = h(g(s))$, where g is some fixed function, is *linear*, hence class C^∞. It follows that F is as smooth as the function f.

We will apply the implicit function theorem to F. Note first that $F(0,0)(s) = y(T,s,0,0)$. For $(h,\delta) = (0,0)$, $\dot{w} = Df(\phi_t(p))w$ and $w(0,s,0,0) = sg(p)$. Hence, we have $w(T,s,0,0) = sb(T,p)g(p)$; that is, $x(T,s,0,0) = sb(T,p)$ and $y(T,s,0,0) = 0$. So, $F(0,0) = 0$. Next, we compute $F_h(0,0)$. For this, we set $\delta = 0$ and differentiate the function $\mathcal{F} : \mathcal{B} \to \mathcal{E}$ given by $\mathcal{F}(h)(s) = y(T,s,h,0) - h(x(T,s,h,0))$ at $h = 0$. In this case, we have that $\dot{w} = Df(\phi_t(p))w$ and $w(0,s,h,0) = h(s)f(p) + sg(p)$. Hence, $w(T,s,h,0) = h(s)f(p) + sb(T,p)g(p)$; that is, $x(T,s,0,0) = sb(T,p)$, $y(T,s,0,0) = h(s)$, and

$$\mathcal{F}(h)(s) = h(s) - h(b(T,p)s).$$

Because $\mathcal{F}$ is *linear*, we have that $F_h(0,0) : \mathcal{E} \to \mathcal{E}$ is given by $F_h(0,0) = I - U$, where I is the identity transformation and $U : \mathcal{E} \to \mathcal{E}$ is the linear transformation given by $U(k)(s) = k(b(T,p)s)$. For $\|k\|_{\mathcal{E}} = 1$ and $s \neq 0$, we have

$$|U(k)'(s)| = b(T,p)|k'(b(T,p)s)|.$$

It follows that $\|U\|_{\mathcal{E}} = b(T,p) < 1$ and $I - U$ is invertible ($(I-U)^{-1} = \sum_{n=0}^{\infty} U^n$). By the implicit function theorem, there is an open subinterval K of J with $0 \in K$ and a smooth function $\beta : K \to \mathcal{B}$ such that $K(0) = 0$ and $F(K(\delta),\delta) \equiv 0$. For $\delta \in K$ and $\delta > 0$, $K(\delta) \in \mathcal{E}$ is a function whose graph, the image of the curve $s \mapsto \delta K(\delta)(s)f(p) + \delta s g(p)$ defined for $s \in (-1,1)$, is the desired ϕ_T-invariant transverse section at $p \in \Gamma$. $\square$

Note that a ϕ_T-invariant normal foliation is necessarily tangent to an invariant normal bundle at Γ. For hyperbolic limit cycles, there is no ambiguity due to the uniqueness of the invariant normal bundle. But, for nonhyperbolic limit cycles with $a(T,p) = 0$, there are infinitely many invariant normal bundles (see Proposition 2.3). By our main result in this section (Theorem 2.8), if Γ is a nonhyperbolic limit cycle and $a(T,p) = 0$, then there is a unique ϕ_T-invariant normal foliation under some generic conditions. Therefore, only one invariant normal bundle can be realized by a ϕ_T-invariant normal foliation.

We will define a function $Q : \Gamma \to \mathbb{R}^2$, which depends on the second derivative of the underlying vector field along Γ; it is used to identify the unique invariant normal bundle whose fibers are tangent to the leaves of the ϕ_T-invariant normal foliation at Γ. Indeed, let us denote the usual inner product of $u, v \in \mathbb{R}^2$ by $\langle u, v\rangle$ and define

$$Q(p) := \int_0^T D\phi_{-t}(\phi_t(p)) D^2 f(\phi_t(p))(D\phi_t(p) f^\perp(p), D\phi_t(p) f^\perp(p))\, dt. \tag{15}$$

Proposition 2.6. *Suppose that υ and ω are real numbers, $p \in \Gamma$, and $u(t,\zeta) := \phi_t(\zeta)$ for $\zeta \in \mathbb{R}^2$. If Γ is not hyperbolic and the derivative of the return-time map vanishes at Γ, then*

$$u_{\zeta\zeta}(T,p)(\upsilon f(p) + \omega f^\perp(p), \upsilon f(p) + \omega f^\perp(p)) = \omega^2 u_{\zeta\zeta}(T,p)(f^\perp(p), f^\perp(p))$$

and

$$Q(p) = u_{\zeta\zeta}(T,p)(f^\perp(p), f^\perp(p)). \tag{16}$$

Proof. Since $\phi_t(p) \in \Gamma$, $a(T, \phi_t(p)) = 0$ and $b(T, \phi_t(p)) = 1$, we have the identity $u_\zeta(T, \phi_t(p)) = I$. Hence, if $J \subseteq \mathbb{R}$ is a sufficiently small open interval containing the origin and $\gamma : J \to \mathbb{R}^2$ is a smooth function such that $\gamma(0) = 0$, then $u_\zeta(T, \phi_s(p))\gamma(s) = \gamma(s)$. By differentiating with respect to s at $s = 0$, it follows that $u_{\zeta\zeta}(T,p)(f(p), \dot\gamma(0)) = (u_\zeta(T,p) - I)\dot\gamma(0) = 0$. In particular $u_{\zeta\zeta}(T,p)(f(p), f(p)) = 0$ and $u_{\zeta\zeta}(T,p)(f(p), f^\perp(p)) = 0$. On the other hand, since $u(0,\zeta) = \zeta$, $\dot u = f(u)$, $u_\zeta(0,\zeta) = I$, and $\dot u_\zeta = Df(u)u_\zeta$. The solution of the second-order variational equation $\dot u_{\zeta\zeta} = Df(u)u_{\zeta\zeta} + D^2 f(u)(u_\zeta, u_\zeta)$ yields

$$u_{\zeta\zeta}(T,p) = \int_0^T D\phi_{-t}(\phi_t(p)) D^2 f(\phi_t(p))(D\phi_t(p)(\cdot), D\phi_t(p)(\cdot))\, dt.$$

The desired result follows from the bilinearity of second derivatives and the definition of Q. ☐

Let $\tilde J \subseteq \mathbb{R}$ be an open interval containing the origin and $\sigma : \tilde J \to \mathbb{R}^2$ a C^2 function such that $\sigma(0) = p$ and σ is transverse to Γ at p. There is a subinterval $J \subseteq \tilde J$ containing the origin such that the image Σ of σ restricted to J is a Poincaré section with Poincaré map P. Using the coordinate along Σ given by its parameterization, the local coordinate representation of P, which we again denote by P, is given by

$$\sigma(P(s)) = u(T(\sigma(s)), \sigma(s)), \tag{17}$$

where $u(t,\zeta) := \phi_t(\zeta)$. Of course, in these coordinates $P(0) = 0$. Also, if Γ is not hyperbolic, then $P'(0) = 1$.

Proposition 2.7. *Suppose that Σ is a Poincaré section at $p \in \Gamma$ with Poincaré map P, and a coordinate on Σ is defined by $\sigma : J \to \mathbb{R}^2$ such that $J \subset \mathbb{R}$ is an open interval containing the origin and $\sigma(0) = p$. If Γ is not hyperbolic and the derivative of the return-time map vanishes at Γ, then (in the local coordinate given by σ)*

$$P''(0) = \frac{\langle \dot\sigma(0), f^{\perp}(p)\rangle}{|f(p)|^4}\langle Q(p), f^{\perp}(p)\rangle. \tag{18}$$

Proof. By differentiation in equation (17), we find that

$$\dot\sigma(P(s))P'(s) = f(u(T(\sigma(s)),\sigma(s)))(dT(\sigma(s))\dot\sigma(s)) + u_{\zeta}(T(\sigma(s)),\sigma(s))\dot\sigma(s).$$

By using the hypotheses, differentiating again with respect to s at $s = 0$ and rearranging the resulting equation, we have that

$$\dot\sigma(0)P''(0) = d^2T(p)(\dot\sigma(0),\dot\sigma(0))f(p) + u_{\zeta\zeta}(T,p)(\dot\sigma(0),\dot\sigma(0)).$$

Finally, by taking the inner product of both sides of the last equation with respect to $f^{\perp}(p)$, using the identity

$$\dot\sigma(0) = \frac{1}{|f(p)|^2}(\langle\dot\sigma(0), f(p)\rangle f(p) + \langle\dot\sigma(0), f^{\perp}(p)\rangle f^{\perp}(p)),$$

and using Proposition 2.6, we obtain formula (18). ☐

Theorem 2.8. *Suppose that f is C^2 and $p \in \Gamma$. If Γ is not hyperbolic, the derivative of the return-time map vanishes at Γ, and the second derivative of the Poincaré map P at Γ is not zero, then Γ has a ϕ_T-invariant normal foliation whose leaves are tangent to the leaves of the normal bundle given by $D\phi_t(p)g(p)$ for $t \in [0,T)$, where $g(p) = A(p)f(p) + f^{\perp}(p)$ and $A(p) = \langle Q(p), f(p)\rangle / \langle Q(p), f^{\perp}(p)\rangle$. In particular, Γ is isochronous.*

Proof. Under the hypotheses of the theorem, if Σ is a Poincaré section with coordinate s such that $s = 0$ is the coordinate of $p = \Sigma \cap \Gamma$, then $P(s) = s + (P''(0)/2!)s^2 + O(s^3)$ with $P''(0) \neq 0$; therefore, Γ is semistable.

We will consider the stable and unstable 'sides' of Γ separately. In each case there is an invariant normal foliation and these foliations match to first-order at their base points.

Let $\mathcal{C}$ denote the Banach space consisting of functions $h \in C([0,1],\mathbb{R})$ such that $h(0) = 0$ and $\|h\|_{\mathcal{C}} := \sup_{s\in(0,1]} |h(s)|/|s|^3 < \infty$. Also, let $\mathcal{E}$ denote the Banach space consisting of functions $h \in C^1([0,1],\mathbb{R})$ such that $h(0)=0$, $h'(0)=0$, and $\|h\|_{\mathcal{E}} := \sup_{s\in(0,1]} |h'(s)|/|s| < \infty$. Here, $h'(0)$ is the right-hand derivative. Also, let $g(p)$ denote the vector defined in the statement of the theorem.

We will consider the stable side of Γ. For simplicity, let us assume that $f^{\perp}(p)$ is in the direction of the stable side; that is, the curve $\sigma(s) := p + sf^{\perp}(p)$ has image in the stable set of Γ for sufficiently small $s > 0$. Moreover, with respect to the corresponding Poincaré section given by the (restricted) image of σ, the corresponding Poincaré map P is such that $P''(0) < 0$.

As in the proof of Theorem 2.5, we will show that there is a smooth function $F : B \times J \to \mathcal{C}$, with suitable B and J, given by $F(h,\delta) = y(T,s,h,\delta) - h(x(T,s,h,\delta))$, where x and y are defined as before, but here $s \in [0,1]$. To show that for suitable B and J, $x(T,s,h,\delta)$ is in the domain of h, we will expand $w(T,s,h,\delta)$ with respect to δ at $\delta = 0$.

Using the variational initial value problem (at $\delta = 0$)

$$\dot{w}_\delta = Df(\phi_t(p))w_\delta + D^2 f(\phi_t(p))(w(t),w(t)), \quad w_\delta(t,s,h,0) = 0,$$

where $w(t,s,h,0) = D\phi_t(p)(h(s)f(p) + sg(p))$ (see equation (13)). Since $a(T,p) = 0$ and $b(T,p) = 1$, we have that $D\phi_T(p) = I$, where I is the identity on $\mathbb{R}^2$; hence, the variation of parameters formula yields the solution

$$w_\delta(T,s,h,0) = \int_0^T D\phi_{-t}(\phi_t(p))D^2 f(\phi_t(p))(w(t,s,h,0),w(t,s,h,0))\,dt.$$

By Proposition 2.6 and the representation of $w(t,s,h,0)$ relative to $f(p)$ and $g(p)$, and the representation of $g(p)$ relative to $f(p)$ and $f^{\perp}(p)$, we find that

$$w_\delta(T,s,h,0) = s^2 \int_0^T D\phi_{-t}(\phi_t(p))D^2 f(\phi_t(p))(D\phi_t(p)f^{\perp}(p), D\phi_t(p)f^{\perp}(p))\,dt.$$

By resolving w_δ into components and using equation (18), we have the equalities

$$\begin{aligned}
w_\delta(T,s,h,0) &= s^2\frac{1}{|f(p)|^2}(\langle Q(p), f(p)\rangle f(p) + \langle Q(p), f^\perp(p)\rangle f^\perp(p)) \\
&= s^2\frac{1}{|f(p)|^2}[(\langle Q(p), f(p)\rangle - \langle Q(p), f^\perp(p)\rangle A(p) \\
&\quad + \langle Q(p), f^\perp(p)\rangle A(p)) f(p) \\
&\quad + \langle Q(p), f^\perp(p)\rangle f^\perp(p)] \\
&= s^2\frac{1}{|f(p)|^2}\langle Q(p), f^\perp(p)\rangle g(p) \\
&= s^2 P''(0) g(p);
\end{aligned}$$

therefore, if we restrict h to a bounded subset of $\mathcal{E}$, then

$$w(T,s,h,0) = h(s)f(p) + sg(p) + \delta s^2 P''(0)g(p) + O(\delta^2)$$

and

$$x(T,s,h,0) = s + \delta P''(0)s^2 + O(\delta^2), \qquad y(T,s,h,0) = h(s) + O(\delta^2). \tag{19}$$

For sufficiently small δ, we have that $0 \le x(T,s,h,0) \le 1$, as required. Also, using these estimates it follows that the range of F is in $\mathcal{C}$.

To complete the proof, it suffices to show (as in the proof of Theorem 2.5) that there is some $h \in \mathcal{E}$ and $\delta > 0$ such that $F(h,\delta) = 0$.

Since $D\phi_T(p) = I$, it follows immediately that $F(h,0) \equiv 0$. By Taylor's theorem (with the integral form of the remainder) there is a C^2-function $R : B \times J \to \mathcal{C}$ such that $R(h,0) \equiv 0$ and $F(h,\delta) = \delta(F_\delta(h,0) + R(h,\delta))$. (Here, we may have to take open subsets of the original sets B and J. Also, we are identifying $L(\mathbb{R},\mathcal{E})$ with $\mathcal{E}$.) Thus, it suffices to find k and $\gamma > 0$ such that $\mathcal{F}(k,\gamma) = 0$ for the function $\mathcal{F} : B \times J \to \mathcal{C}$ given by

$$\mathcal{F}(h,\delta) = F_\delta(h,0) + R(h,\delta).$$

By an application of the implicit function theorem applied to $\mathcal{F}$, it suffices to find a simple zero of the function $h \mapsto F_\delta(h,0)$.

A direct computation yields the identities

$$\begin{aligned}
F_\delta(h,0) &= y_\delta(T,s,h,0) - h'(x(T,s,h,0))x_\delta(T,h,s,0) \\
&= y_\delta(T,s,h,0) - h'(s)x_\delta(T,h,s,0).
\end{aligned}$$

Using the partial derivatives x_δ and y_δ given in display (19), we obtain the representation

$$F_\delta(h,0)(s) = -\frac{1}{|f(p)|^2}\langle Q(p), f^\perp(p)\rangle s^2 h'(s).$$

Finally, we have $F_\delta(0,0)=0$ and

$$(F_{\delta h}(0,0)k)(s) = -\frac{1}{|f(p)|^2}\langle Q(p), f^\perp(p)\rangle s^2 k'(s).$$

The corresponding linear transformation $F_{\delta h}(0,0) : \mathcal{E} \to \mathcal{C}$ is bounded and has a bounded inverse. In fact, its inverse $\Psi : \mathcal{C} \to \mathcal{E}$ is given by

$$\Psi(k)(s) = \frac{1}{c}\int_0^s K(t)\,dt,$$

where $c := -\langle Q(p), f^\perp(p)\rangle/|f(p)|^2$ and K is defined by $K(s) = k(s)/s^2$ for $s \neq 0$ and $K(0) = 0$. Hence, $h = 0$ is a simple zero of $h \mapsto F_\delta(h,0)$, as required.

Note that the ϕ_T-invariant curve given by $s \mapsto p + \delta h(s) f(p) + \delta s g(p)$ is tangent to $g(p)$ at p. This will be true for the corresponding curve that lies in the unstable set of Γ. These curves together form a C^1, ϕ_T-invariant leaf at p, which can be made a leaf of a ϕ_T-invariant foliation obtained by pushing this leaf to every point on Γ using the flow ϕ_t. $\square$

3 nonisochronous Limit Cycles

By Theorem 2.8, if Γ is a nonhyperbolic limit cycle, $p \in \Gamma$, and $P''(p) \neq 0$, then $a(T,p) = 0$ is a *sufficient* condition for Γ to be isochronous. We will show that this condition is also *necessary.*

Consider the planar differential equation

$$\dot{u} = f(u), \tag{20}$$

where $u = (x,y) \in \mathbb{R}^2$ and f is class C^r with $r \geq 2$. As before, let us suppose that this differential equation has a limit cycle Γ and $p \in \Gamma$. Also, we will assume that Σ is a transverse section at p with associated return-time map τ and Poincaré map P. In particular, $T := \tau(p)$ is the period of Γ.

Theorem 3.1. *Suppose that* Γ *is a limit cycle of the differential equation (20) and* $p \in \Gamma$. *If* $a(T, p) \neq 0$, $P'(p) = 1$ *(or equivalently* $b(T, P) = 1$*), and* $P''(p) \neq 0$, *then* Γ *is not isochronous. In fact, there are no points (not on* Γ*) in phase with a point on* Γ.

Our proof of Theorem 3.1 requires some precise estimates for the asymptotic behavior of the orbits of the Poincaré map P on Σ. We will prove some lemmas to obtain these estimates before presenting the proof of the theorem.

Let us consider a parameterization of the section Σ so that P is at the origin in the corresponding local coordinate σ. In the context of Theorem 3.1, the local representation of P near the origin is given by

$$P(\sigma) = \sigma + c\sigma^2 + o(\sigma^2), \tag{21}$$

where c is a nonzero constant.

Lemma 3.2. *If* $0 < z_1 < 1$ *and (for* $n \geq 2$*) the real number* z_{n+1} *is defined recursively by* $z_{n+1} = z_n - z_n^2$, *then the sequence* $\{z_n\}_{n=1}^{\infty}$ *converges monotonically to zero. If, in addition,* $z_1 < 1/2$ *and* $\gamma \in (1, 2)$, *then there exists an integer* $N > 1$ *such that,*

$$z_n < \frac{1}{n} \text{ for all } n \geq 1, \quad \text{and } z_{n-N+1} > \frac{1}{n} - \frac{1}{n^{\gamma}} \text{ for all } n \geq N.$$

Proof. The proof of the first statement of the lemma is left to the reader. In effect, $z = 0$ is a globally attracting fixed point for the quadratic map $z \mapsto z - z^2$ restricted to the interval $[0, 1]$.

We will determine the asymptotic behavior of the sequence $\{z_n\}_{n=1}^{\infty}$ in case $0 < z_1 < 1/2$.

Note that

$$\frac{1}{n+1} = \frac{1}{n} - \frac{1}{n^2} + \frac{1}{n^2(n+1)}.$$

Hence,

$$\begin{aligned} z_{n+1} - \frac{1}{n+1} &= \left(z_n - \frac{1}{n}\right)\left(1 - z_n - \frac{1}{n}\right) - \frac{1}{n^2(n+1)} \\ &< \left(z_n - \frac{1}{n}\right)\left(1 - z_n - \frac{1}{n}\right). \end{aligned}$$

Since $0 < z_2 < 1/2$, it follows that $z_n < 1/n$ for $n \geq 3$. This completes the proof of the first inequality in the second statement of the lemma.

To prove the second inequality in the second statement of the lemma, we will use the inequality

$$-\frac{1}{n^\gamma} + \frac{1}{(n+1)^\gamma} \geq -\frac{\gamma}{n^{1+\gamma}}, \tag{22}$$

which holds for $\gamma > 0$ and $n \geq 1$. To prove it, let $h(x) = x^{-\gamma}$ and note that the desired inequality is

$$h(n+1) - h(n) \geq h'(n).$$

By applying the mean value theorem to h, there is some number $\beta \in (0,1)$ such that

$$h(n+1) - h(n) = h'(n+\beta) \geq h'(n),$$

as required.

For $\gamma \in (1,2)$, let $a_n := \frac{1}{n} - \frac{1}{n^\gamma}$. Using the inequality (22), we have that $a_{n+1} = a_n - a_n^2 - E_n$, where

$$\begin{aligned} E_n &= a_n - a_{n+1} - a_n^2 \\ &= \frac{1}{n} - \frac{1}{n^\gamma} - \frac{1}{n+1} + \frac{1}{(n+1)^\gamma} - \left(\frac{1}{n} - \frac{1}{n^\gamma}\right)^2 \\ &= \frac{2}{n^{1+\gamma}} - \frac{1}{n^\gamma} + \frac{1}{(n+1)^\gamma} - \frac{1}{n^2(n+1)} - \frac{1}{n^{2\gamma}} \\ &\geq \frac{2}{n^{1+\gamma}} - \frac{\gamma}{n^{1+\gamma}} - \frac{1}{n^2(n+1)} - \frac{1}{n^{2\gamma}} \\ &= \frac{2-\gamma}{n^{1+\gamma}} - \frac{1}{n^2(n+1)} - \frac{1}{n^{2\gamma}}. \end{aligned}$$

Since $1 < \gamma < 2$, there exists $N' > 1$ such that $E_n > 0$ for $n \geq N'$. Choose $N \geq N'$ so that $z_1 > a_N$. Let $y_n := z_{n-N+1}$ for $n \geq N$. Then, for $n \geq N$,

$$\begin{aligned} y_{n+1} - a_{n+1} &= (y_n - a_n)(1 - y_n - a_n) + E_n \\ &\geq (y_n - a_n)(1 - y_n - a_n). \end{aligned}$$

Recall that z_n is monotonically decreasing to zero and note that $a_n < 1/2$ for all $n \geq N$. Since $a_N < y_N = z_1 < 1/2$, it follows that $y_n < 1/2$ for all $n \geq N$ and by induction $y_n > a_n$ for all $n \geq N$. Equivalently, $z_{n-N+1} > a_n$ for all $n \geq N$, as required. □

Corollary 3.3. *Suppose that* $c < 0$ *in the local representation of the Poincaré map* P *defined in display (21). If* $\sigma_0 > 0$ *and* $\sigma_n := P^n(\sigma_0)$ *for* $n \geq 1$*, then* $\lim_{n\to\infty} \sigma_n = 0$ *and* $\sigma_n = O(1/n)$*; in particular, the series* $\sum_{n=0}^{\infty} \sigma_n$ *diverges.*

Proof. From display (21), we have that $P(\sigma) = \sigma + c\sigma^2 + o(\sigma^2)$; therefore, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sigma + (c - \epsilon)\sigma^2 < P(\sigma) < \sigma + (c + \epsilon)\sigma^2 \tag{23}$$

whenever $0 < \sigma < \delta$. Since we are interested in the asymptotic behavior of the sequence $\{\sigma_n\}_{n=1}^{\infty}$, we will assume (without loss of generality) that $0 < \sigma_n < \delta$ for all n.

Let $\rho_n := (|c| + \epsilon)\sigma_n$. By the inequality (23), $\sigma_{n+1} > \sigma_n - (|c| + \epsilon)\sigma_n^2$. Hence, we have that $\rho_{n+1} > \rho_n - \rho_n^2$. In case $1/2 > \rho_1 \geq z_1$, where z_n is defined in Lemma 3.2, the inequalities $\rho_n \geq z_n$ for $n \geq 1$ are proved by the same argument used in the proof Lemma 3.2. The estimate in Lemma 3.2 implies $\rho_{n-N+1} \geq 1/n - 1/n^\gamma$ for all $n \geq N$, where $\gamma \in (1, 2)$ and $N \geq 1$ are as in Lemma 3.2. It follows that, for all $n \geq N$, $\sigma_{n-N+1} \geq C_1(1/n - 1/n^\gamma)$ for some positive number C_1. A similar method can be used to prove that there is a positive constant C_2 such that, for $n \geq 1$, $\sigma_n \leq C_2/n$. The statement of the corollary follows easily from these estimates. ☐

Proof of Theorem 3.1. Suppose that $u_0 \in \mathbb{R}^2 \setminus \Gamma$ has an asymptotic phase. We will show that this assumption leads to a contradiction.

Without loss of generality, we will assume that Γ is the unit circle, $p = (1, 0)$, Γ is the orbit of the solution $\phi_t(p) = (\cos(2\pi t), \sin(2\pi t))$ with period $T = 1$, the point u_0 is in phase with $p = (1, 0)$, and the section Σ at p is on the x-axis. Also, for definiteness, we will assume that $a(T, p) > 0$ and $P''(p) < 0$. In this case, Γ is attracting on the outside; that is, Γ is attracting in a neighborhood of points q in the plane where $|q| > 1$. Hence, we will assume that $|u_0| > 1$. The proofs for the other cases are similar.

Let $u_n := \phi_n(u_0)$ for $n \geq 0$. Since u_0 is in phase with p, we have that $\lim_{n\to\infty} u_n = p$. There is an open set $U \subset \mathbb{R}^2$ with $p \in U$ such that if $q \in U$, then $\phi_t(q) \notin U$ for some $t \in \mathbb{R}$ with $|t| < 1/2$. There is an open subsection Σ' of Σ that contains p and is contained in U. Also, there is some $\beta > 0$ such that $\beta < 1/2$ and the flow box $V := \{\phi_t(\sigma) : |t| < \beta \text{ and } \sigma \in \Sigma'\}$ is contained in U. Choose a positive integer N such that $u_n \in V$ for all $n \geq N$ and note that u_N is in phase with p.

Define $w_n := u_{N+n}$ for each integer $n \geq 0$. There is a number δ_0, with $|\delta_0| < \beta$ and a point $v_0 = (x_0, 0) \in \Sigma'$ such that $\phi_{\delta_0}(v_0) = w_0$. Using the point

v_0, we define $v_{n+1} = P(v_n)$ and note that $P(v_n) = \phi_{\tau_n}(v_n)$, where $\tau_n := \tau(v_n)$ and τ is the return-time map on Σ.

For $n \geq 1$, since v_n and w_n are in V, there is a number δ_n, with $|\delta_n| < \beta$, such that $\phi_{\delta_n}(v_n) = w_n$. Using this relation, together with the bound on the sequence $\{\delta_n\}_{n=0}^{\infty}$, the structure of the flow box V, and the assumptions that the sequences $\{w_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ both converge to p, it follows that $\lim_{n\to\infty} \delta_n = 0$. This fact can be proved by assuming that $\{\delta_n\}_{n=0}^{\infty}$ does not converge to zero, in which case there is a subsequence of $\{\phi_{\delta_n}(v_n)\}_{n=0}^{\infty}$ that does not converge to p.

Since $\phi_1(w_{n-1}) = w_n$ and $\phi_{\delta_n}(v_n) = w_n$, it follows that $\phi_{1+\delta_n}(v_n) = w_{n+1} = \phi_{\delta_{n+1}}(v_{n+1})$; or equivalently, $v_{n+1} = \phi_{1+\delta_n-\delta_{n+1}}(v_n)$. In other words, $\tau_n = 1 + \delta_n - \delta_{n+1}$. Hence, we have

$$\sum_{j=0}^{n-1} \tau_j = n + \delta_0 - \delta_n,$$

and therefore, $\lim_{n\to\infty}(\sum_{j=0}^{n-1} \tau_j - n) = \delta_0 < \infty$.

On the other hand, since $a(T,p) > 0$, we have $\tau'(p) > 0$ due to Proposition 2.2. Hence, there is a number $\lambda > 0$ such that ,

$$\tau_j = \tau(v_j) = \tau(p) + \tau'(p)(x_j - 1) + o(|x_j - 1|) \geq 1 + \lambda(x_j - 1)$$

for all $j \geq 0$. Therefore, we have that

$$\sum_{j=0}^{n-1} \tau_j - n \geq \lambda \sum_{j=0}^{n-1}(x_j - 1).$$

Note that $x_j - 1 = \sigma_j$ in the local representation of the Poincaré section. By Corollary 3.3, the summation on the right diverges as $n \to \infty$. The contradiction completes the proof. $\square$

Remark 3.4. *Using our methods, it is possible to prove results for more degenerate limit cycles. In fact, if* $m \geq 2$ *and the Poincaré map* P *is given by*

$$P(\sigma) = \sigma + c\sigma^m + o(\sigma^m) \;\; \textit{for some} \;\; c \neq 0$$

then Γ *is isochronous if and only if*

$$\tau'(p) = \tau''(p) = \cdots = \tau^{m-1}(p) = 0.$$

By Theorems 2.8 and 3.1, a nonhyperbolic limit cycle Γ of a C^2 system with a corresponding Poincaré map, which has a nonvanishing second derivative at Γ, is isochronous if and only if $a(T,p) = 0$ for $p \in \Gamma$. The generic condition (that is, a Poincaré map has a nonvanishing second derivative at Γ) can only be imposed if the system is at least C^2. This smoothness requirement is essential. In fact, our analysis of the following family of systems in Example 1 shows that, for $\alpha \in (0,1)$, there is a $C^{1+\alpha}$ system with a nonhyperbolic *isochronous* limit cycle Γ such that $a(T,p) \neq 0$. Also, we show that Theorem 3.1 can be verified in an example by a direct and simple computation.

Example 1. Consider the planar system

$$\dot{r} = -|r-1|^{1+\alpha}, \qquad \dot{\theta} = 2\pi + (r-1), \tag{24}$$

where (r,θ) are polar coordinates and $\alpha > 0$ is a parameter. We note that the unit circle, given by $r = 1$, is a periodic orbit Γ for system (24) with period $T = 1$. Also, since $\alpha > 0$, this orbit is not hyperbolic. In fact, it is semi-stable, attracting in the region where $r > 1$ and repelling in the region where $r < 1$.

In rectangular coordinates, system (24) is given by

$$\begin{aligned} \dot{x} &= -|\sqrt{x^2+y^2}-1|^{1+\alpha} \cdot \frac{x}{\sqrt{x^2+y^2}} - y\left(2\pi + \sqrt{x^2+y^2} - 1\right), \\ \dot{y} &= -|\sqrt{x^2+y^2}-1|^{1+\alpha} \cdot \frac{y}{\sqrt{x^2+y^2}} + x\left(2\pi + \sqrt{x^2+y^2} - 1\right). \end{aligned} \tag{25}$$

We note that this system is $C^{1+\alpha}$ in $\mathbb{R}^2 \setminus \{(0,0)\}$ for $\alpha > 0$, and it is C^∞ for α a positive integer. Also, it has the periodic solution $\phi_t(p) = (\cos(2\pi t), \sin(2\pi t))$, where $p = (1,0)$.

Lemma 3.5. *If $\alpha > 0$ in system (24), then $b(t,p) = 1$ for all t and $a(T,p) = -1$.*

Proof. On the periodic solution,

$$\dot{x} = -2\pi\sin(2\pi t),\ \ddot{x} = -4\pi^2\cos(2\pi t),\ \dot{y} = 2\pi\cos(2\pi t),\ \ddot{x} = -4\pi^2\sin(2\pi t).$$

Hence, $\kappa(t,p) = 1$ (the curvature of the unit circle) and $2\kappa(t,p)|f(\phi_t(p))| = 4\pi$. By direct computation, $\operatorname{curl} f(\phi_t(p)) = 4\pi + 1$ and $\operatorname{div} f(\phi_t(p)) = 0$. Using these equations together with formulas (2) and (3), it follows that $b(t,p) = 1$ for all t and $a(T,p) = -1$. $\square$

Let $t \mapsto (r(t, r_0, \theta_0), \theta(t, r_0, \theta_0))$ denote the solution of the differential equation (24) such that $r(0, r_0, \theta_0) = r_0$ and $\theta(0, r_0, \theta_0) = \theta_0$.

Proposition 3.6. *If $0 < \alpha < 1$ in system (24), then Γ is isochronous. More precisely, if $r_0 > 1$, then*

$$\lim_{t\to\infty} (r(t, r_0, \theta_0), \theta(t, r_0, \theta_0)) = (1, \theta_\infty + 2\pi t),$$

where

$$\theta_\infty := \theta_0 - \frac{(r_0 - 1)^{1-\alpha}}{\alpha - 1}. \tag{26}$$

If $\alpha \geq 1$, then Γ is not isochronous. In fact, there are no points (not on Γ) that are in phase with a point on Γ.

Proof. The solution of system (24) for $r_0 > 1$, $\alpha > 0$, and $\alpha \neq 1$ is given by

$$\begin{aligned} r(t, r_0, \theta_0) &= 1 + \big((r_0 - 1)^{-\alpha} + \alpha t\big)^{-1/\alpha}, \\ \theta(t, r_0, \theta_0) &= \theta_0 + 2\pi t + \int_0^t (r(s; r_0, \theta_0) - 1)\, ds \\ &= \theta_0 + 2\pi t + \frac{1}{\alpha - 1}\big((r_0 - 1)^{-\alpha} + \alpha t\big)^{(\alpha-1)/\alpha} - \frac{(r_0 - 1)^{1-\alpha}}{\alpha - 1}. \end{aligned}$$

For $\alpha = 1$, the solution is

$$\begin{aligned} r(t, r_0, \theta_0) &= 1 + \big((r_0 - 1)^{-1} + t\big)^{-1}, \\ \theta(t, r_0, \theta_0) &= \theta_0 + 2\pi t + \int_0^t (r(s; r_0, \theta_0) - 1)\, ds \\ &= \theta_0 + 2\pi t + \ln\big((r_0 - 1)^{-1} + t\big) + \ln(r_0 - 1). \end{aligned}$$

Clearly, $\lim_{t\to\infty} r(t, r_0, \theta_0) = 1$. If $0 < \alpha < 1$, then

$$\lim_{t\to\infty} (\theta(t, r_0, \theta_0) - (\theta_\infty + 2\pi t)) = \lim_{t\to\infty} \frac{1}{\alpha - 1}\big((r_0 - 1)^{-\alpha} + \alpha t\big)^{(\alpha-1)/\alpha} = 0.$$

On the other hand, if $\alpha \geq 1$ and $(1, \hat{\theta})$ is on the periodic orbit, then

$$\lim_{t\to\infty} (\theta(t, r_0, \theta_0) - (\hat{\theta} + 2\pi t)) = \infty.$$

Hence, the point with polar coordinates (r_0, θ_0) does not have an asymptotic phase. □

By Proposition 3.6, if $0 < \alpha < 1$ in system (24), then there is a ϕ_T-invariant foliation for the periodic orbit Γ. The leaf through the point on Γ with polar coordinates $(1, \theta_\infty)$ is given by the polar equation (26). An easy computation shows that each leaf is *tangent* to Γ at $(1, \theta_\infty)$. Thus, for this example, $a(T, p) \neq 0$ and there is a ϕ_T-invariant foliation, but the foliation is not a ϕ_T-invariant *normal* foliation.

We will construct C^∞ planar systems each with a nonhyperbolic limit cycle Γ such that $\tau^{(n)}(p) = 0$ for all $n \geq 1$ and $P^{(n)}(p) = 0$ for all $n \geq 2$. The limit cycle Γ is nonisochronous in Example 2; it is isochronous in Example 3.

Example 2. Consider the planar system

$$r' = -\frac{1}{3}(r-1)^4 e^{-|r-1|^{-3}}, \qquad \theta' = 2\pi + e^{-|r-1|^{-3}}, \tag{27}$$

where (r, θ) are polar coordinates. System (27) is C^∞ in an open neighborhood of the unit circle, which is a nonhyperbolic limit cycle with period $T = 1$. For the set of initial conditions (r_0, θ_0) with $r_0 > 1$ for which Γ is attracting, we will show that $\tau^{(n)}(p) = 0$ for $p = (1, 0)$ and all $n \geq 1$, and (r_0, θ_0) is not in phase with a point on Γ. By Remark 3.4 it follows that $P(p) = p$, $P'(p) = 1$, and $P^{(n)}(p) = 0$ for all $n \geq 2$.

Using the change of variable $z = (r-1)^{-1}$, the solution of system (27) with initial condition (r_0, θ_0) is easily found to be

$$\begin{aligned} r(t, r_0, \theta_0) &= 1 + \left(\ln\left(t + e^{(r_0-1)^{-3}}\right)\right)^{-1/3}, \\ \theta(t, r_0, \theta_0) &= \theta_0 + 2\pi t + \ln\left(t + e^{(r_0-1)^{-3}}\right) - (r_0-1)^{-3} \\ &= \theta_0 + 2\pi t + \ln\left(1 + t e^{-(r_0-1)^{-3}}\right). \end{aligned}$$

It is clear that no point (r_0, θ_0) with $r_0 > 1$ has an asymptotic phase. It remains to show that $\tau^{(n)}(1) = 0$ for all $n \geq 1$.

On the Poincaré section $\{(r, 0) : r > 0\}$, the return-time $\tau(r_0)$ is implicitly defined by the equation

$$2\pi\tau(r_0) + \ln\left(1 + \tau(r_0) e^{-(r_0-1)^{-3}}\right) = 2\pi. \tag{28}$$

We will show that $\tau^{(n)}(1) = 0$ for all $n \geq 1$ by contradiction. Suppose that $\tau^{(n)}(1) = \alpha \neq 0$ for some $n \geq 1$, and consider the first such nonzero derivative.

Due to the presence of the positive term $\exp\{-|r-1|^{-3}\}$ in the second equation of system (27), we have that $\tau(r_0) < \tau(1) = 1$ for every $r_0 \neq 1$; hence, $\alpha < 0$ and n must be even. Therefore,

$$\tau(r_0) = 1 + \frac{\alpha}{n!}(r_0-1)^n + o((r_0-1)^n) \leq 1 + A(r_0-1)^n \tag{29}$$

for $|r_0 - 1|$ small and some number A such that $\alpha/n! < A < 0$. Set $x = (r_0-1)^n$. Using equations (28) and (29), we have the inequality

$$2A\pi x + \ln\left(1 + Axe^{-x^{-3/n}} + e^{-x^{-3/n}}\right) \geq 0,$$

or equivalently,

$$2A\pi + \frac{1}{x}\ln\left(1 + Axe^{-x^{-3/n}} + e^{-x^{-3/n}}\right) \geq 0.$$

By an application of L'Hopital's rule, the second term on the left-hand side of the last equation approaches zero as $x \to 0$. It follows that $2A\pi \geq 0$, in contradiction to the assumption $A < 0$. Therefore, $\tau^{(n)}(1) = 0$ for all $n \geq 1$.

Example 3. Consider the planar system

$$r' = -\frac{1}{3}(r-1)^4 e^{-|r-1|^{-3}}, \qquad \theta' = 2\pi, \tag{30}$$

where (r, θ) are polar coordinates. The unit circle is a nonhyperbolic limit cycle with period $T = 1$ that is attracting for $r > 1$. Due to the special form of the θ-equation, the asymptotic phase of the point (r_0, θ_0) is $(1, \theta_0)$; hence, the limit cycle is isochronous. As in Example 2, the solution of system (30) with initial condition (r_0, θ_0) with $r_0 > 1$ is given by

$$\begin{aligned} r(t, r_0, \theta_0) &= 1 + \left(\ln\left(t + e^{(r_0-1)^{-3}}\right)\right)^{-1/3}, \\ \theta(t, r_0, \theta_0) &= \theta_0 + 2\pi t. \end{aligned}$$

On the Poincaré section $\{(r, 0) : r > 0\}$, the return-time map is $\tau(r_0) = 1$ for all r_0 and the Poincaré map restricted to $r_0 \geq 1$ is given by

$$P(r_0) = r := 1 + \left(\ln\left(1 + e^{(r_0-1)^{-3}}\right)\right)^{-1/3}.$$

A proof of the equality $P^{(n)}(1) = 0$ for $n \geq 2$ is similar to the proof of the equality $\tau^{(n)}(1) = 0$ for $n \geq 1$ in Example 2.

For a limit cycle in R^n with one center normal direction, our results apply directly to points on the two-dimensional center manifold. It would be interesting to investigate the existence of asymptotic phase for limit cycles whose center manifolds are at least three-dimensional.

References

[1] R. Abraham, J. Marsden and T. Ratiu, *Manifolds, Tensor Analysis, and Applications,* 2nd Ed. (Springer-Verlag, New York, 1988).

[2] B. Aulbach, Behavior of solutions near manifolds of periodic solutions, *J. Diff. Eqs.*, **39**, 1981, 345–377.

[3] B. Aulbach, Invariant manifolds with asymptotic phase, *Nonlinear Analy., Theory, Meth. and Appl.*, **6**, 1982, 817–827.

[4] E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations,* (New York: McGraw-Hill Book Co. Inc. 1955)

[5] C. Chicone, *Ordinary Differential Equations with Applications,* (Springer-Verlag, New York, 1999).

[6] N. Fenichel, Persistence and smoothness of invariant manifolds for flows, *Ind. Univ. Math.*, **21**, 1971, 193–225;

[7] N. Fenichel, Asymptotic stability with rate conditions, *Ind. Univ. Math.*, **23**, 1974, 1109–1137;

[8] N. Fenichel, Asymptotic stability with rate conditions II, *Ind. Univ. Math.*, **26**, 1977, 81–93 .

[9] J. Guckenheimer, Isochrons and phaseless sets, *J. Math. Biol.*, **1**, 1975, 259–273.

[10] J. K. Hale, *Ordinary Differential Equations*, 2nd ed., (Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., 1980).

[11] M. Hirsch, C. Pugh and M. Shub, *Invariant Manifolds,* Lecture Notes in Mathematics, Vol. 583, (Springer-Verlag, New York, 1977).

[12] M. Hirsch and S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra.* (Academic Press, New York, 1974).