Distortion Minimal Morphing I:
The Theory For Stretching

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November 3, 2005

Abstract

We consider the problem of distortion minimal morphing of n-dimensional compact connected oriented smooth manifolds without boundary embedded in $\mathbb{R}^{n+1}$. Distortion involves bending and stretching. In this paper, minimal distortion is defined (with respect to stretching) as the infinitesimal relative change in volume. The existence of minimal distortion diffeomorphisms between diffeomorphic manifolds is proved. A definition of minimal distortion morphing between two isotopic manifolds is given, and the existence of minimal distortion morphs between every pair of isotopic embedded manifolds is proved.

MSC 2000 Classification: 58E99
Key words: minimal morphing, distortion minimal, geometric optimization

1 Introduction

A morph is a transformation between two shapes through a set of intermediate shapes. While there are important applications in manufacturing [7, 12], computer graphics [10, 11], movie making [9], and mesh construction [5, 6], we focus this discussion on the underlying mathematical theory.

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In the research literature available at present, minimal morphing is considered as a numerical problem where a cost functional is minimized over a finite number of intermediate shapes. We introduce a theory of distortion minimal morphing over a continuous family of states in the context of morphs between n-dimensional oriented compact connected smooth manifolds without boundary embedded in $\mathbb{R}^{n+1}$ whose orientations are inherited from the usual orientation of $\mathbb{R}^{n+1}$. Our cost functional measures a total relative change of volume with respect to a family of diffeomorphisms that defines the morph. Our main result is the existence of a distortion minimal morph (with respect to stretching) between every pair of isotopic submanifolds.

A complete theory of minimal morphing would consider minimal morphs with respect to bending and stretching together with algorithms to compute minimal morphs. We give a complete theory of minimal morphing with respect to stretching here; a theory of minimal morphing with respect to bending and an analysis of algorithms for computing distortion minimal morphs will be the subject of future research.

2 Minimal distortion diffeomorphisms

In this section we prove the existence of distortion minimal diffeomorphisms between diffeomorphic n-dimensional oriented manifolds M and N (which are not necessarily embedded in $\mathbb{R}^{n+1}$) with respective volume forms ω_M and ω_N.

Recall that the Jacobian of a diffeomorphism $h : M \to N$ is defined by the equation

$$h^*\omega_N = J(\omega_M, \omega_N)(h)\,\omega_M,$$

where $h^*\omega_N$ denotes the pullback of the volume form ω_N on N by the diffeomorphism h (see [1]). The Jacobian $J(h) := J(\omega_M, \omega_N)(h)$ depends on the diffeomorphism and the volume forms.

The distortion (due to stretching) $\xi(m)$ at $m \in M$, with respect to a diffeomorphism $h : M \to N$, is defined by

$$\xi(m) = \lim_{\varepsilon \to 0} \frac{\left|\int_{h(A_\varepsilon)} \omega_N\right| - \left|\int_{A_\varepsilon} \omega_M\right|}{\left|\int_{A_\varepsilon} \omega_M\right|}, \tag{1}$$

where $A_\varepsilon \subset M$, for $\varepsilon > 0$, is a nested family of (open) neighborhoods of the point $m \in M$ such that $A_\alpha \subseteq A_\beta$ whenever $\alpha > \beta > 0$ and $\cap_{\varepsilon > 0} A_\varepsilon = m$. In other words, the distortion is the infinitesimal relative change of volume with respect to h. We note that a possible choice for the nested sequence of neighborhoods is $A_\varepsilon := \phi^{-1}(B_\varepsilon(q))$, where (U,ϕ) is a local coordinate chart about $m \in M$, $\phi(q) = m$, and $B_\varepsilon(q)$ is the ball in $\mathbb{R}^n$ of radius ε centered at q.

Proposition 2.1. *If $h : M \to N$ is a diffeomorphism and $J(h)$ is the Jacobian of h with respect to the volume forms ω_M on M and ω_N on N, then the corresponding distortion map $\xi : M \to \mathbb{R}$ is given by*

$$\xi(m) = \big|J(h)(m)\big| - 1.$$

In particular, the distortion does not depend on the choice of the nested family of neighborhoods used to define it.

Proof. By the change of variables formula,

$$\int_{h(A_\varepsilon)} \omega_N = \int_{A_\varepsilon} h^* \omega_N.$$

Hence, we have

$$\int_{h(A_\varepsilon)} \omega_N = \int_{A_\varepsilon} J(h) \omega_M.$$

Notice that

$$\lim_{\varepsilon \to 0} \frac{\int_{A_\varepsilon} J(h)\omega_M}{\mathrm{Vol}(A_\varepsilon)} = \big|J(h)(m)\big|.$$

Thus,

$$\xi(m) = \lim_{\varepsilon \to 0} \frac{\int_{A_\varepsilon} J(h)\omega_M - \mathrm{Vol}(A_\varepsilon)}{\mathrm{Vol}(A_\varepsilon)} = \big|J(h)(m)\big| - 1,$$

as required.

The definition of distortion does not depend on the family of nested sets A_ε. Indeed, the choice of these neighborhoods does not change the proof of the first part of the proposition. □

In the following we will denote the set of all C^2 diffeomorphisms between manifolds M and N by $\mathrm{Diff}^2(M,N)$. The *total distortion* functional $\Phi : \mathrm{Diff}^2(M,N) \to \mathbb{R}$, with respect to the oriented manifolds (M,ω_M) and (N,ω_N), is defined by

$$\Phi(h) = \int_M \Big(\big|J(h)(m)\big| - 1\Big)^2 \omega_M. \tag{2}$$

We will establish necessary and sufficient conditions for a diffeomorphism $h : M \to N$ to be a minimum of the functional Φ. Also, we will show that a minimum always exists in $\operatorname{Diff}^2(M,N)$ provided that the manifolds are compact, connected, and without boundary.

As a useful notation, we let $\mathfrak{X}(M)$ denote the set of smooth vector fields on the manifold M. Also, we recall a basic fact from global nonlinear analysis: *$\operatorname{Diff}^2(M,N)$ is a Banach manifold and its tangent space at $h \in \operatorname{Diff}^2(M,N)$ can be identified with $\mathfrak{X}(N)$.* Indeed, an element of $T_h \operatorname{Diff}^2(M,N)$ is an equivalence class of curves $[h_\varepsilon]$, represented by a family of diffeomorphisms h_ε with $h_0 = h$, where two curves passing through h are equivalent if they have the same derivative at h. For each $n \in N$, this family defines a curve $\epsilon \mapsto h_\epsilon(h^{-1}(n))$ in N that passes through n at $\varepsilon = 0$; hence, it defines a vector $Y \in T_nN$ by

$$Y(n) := \frac{d}{d\varepsilon} h_\epsilon(h^{-1}(n))\Big|_{\varepsilon=0}.$$

The vector field $Y \in \mathfrak{X}(N)$ is thus associated with the equivalence class $[h_\varepsilon]$. In fact, the vector field Y does not depend on the choice of the representative of the equivalence class. On the other hand, for $Y \in \mathfrak{X}(N)$ with flow ϕ_t, we associate the curve $h_t = \phi_t \circ h$ in $\operatorname{Diff}^2(M,N)$. The (tangent) equivalence class of this curve is an element in $T_h \operatorname{Diff}^2(M,N)$.

Proposition 2.2 (Euler-Lagrange Equation)**.** *If the diffeomorphism $h \in \operatorname{Diff}^2(M,N)$ is a critical point of the total distortion functional, then*

$$\int_M (|J(h)| - 1) \frac{J(h)}{|J(h)|} (\operatorname{div} Y) \circ h J(h)\, \omega_M = 0 \tag{3}$$

for every $Y \in \mathfrak{X}(N)$.

Proof. Let $h_\varepsilon : (-1,1) \to \operatorname{Diff}^2(M,N)$ be a curve of diffeomorphisms from M to N such that $h_0 = h$. By definition, $h \in \operatorname{Diff}^2(M,N)$ is a critical point of the functional $\Phi(h)$, if $\frac{d}{dt}\Phi(h_t)\Big|_{t=0} = 0$. This derivative is easily computed to be

$$2\int_M (|J(h)| - 1) \frac{J(h)}{|J(h)|} \frac{d}{dt}\big(J(h_t)\omega_M\big)\Big|_{t=0} = 0.$$

Moreover, using the calculus of differential forms (see [1] and note in particular that L_Y is used to denote the Lie derivative in the direction of the vector field Y), we have that for $h^t = \psi_t \circ h$, where ψ_t is the flow of $Y \in \mathfrak{X}(N)$,

$$\begin{aligned}\frac{d}{dt}(J(\psi_t \circ h)\omega_M)\Big|_{t=0} &= \frac{d}{dt}\big((\psi_t \circ h)^* \omega_N\big)\Big|_{t=0} \\ &= h^* \frac{d}{dt}(\psi_t^* \omega_N)\Big|_{t=0} \\ &= h^* \psi_t^* L_Y \omega_N\big|_{t=0} \\ &= h^* L_Y \omega_N \\ &= h^*(\operatorname{div} Y \omega_N) \\ &= (\operatorname{div} Y) \circ h J(h) \omega_M,\end{aligned}$$

as required. □

Proposition 2.3. *Suppose that M and N are smooth connected compact orientable manifolds without boundary. A C^2 diffeomorphism $h : M \to N$ satisfies the Euler-Lagrange equation for the total distortion functional Φ if and only if $J(h)$ is constant.*

Proof. In case $J(h)$ is constant it suffices to show the equality

$$\int_M ((\operatorname{div} Y) \circ h J(h)\, \omega_M = 0.$$

In fact, using Stokes's theorem, we have

$$\begin{aligned}\int_M ((\operatorname{div} Y) \circ h J(h)\, \omega_M &= \int_{h(M)} (\operatorname{div} Y) \omega_N \\ &= \int_N L_Y \omega_N \\ &= \int_N d i_Y \omega_N \\ &= \int_{\partial N} i_Y \omega_N = 0,\end{aligned}$$

as required.

For the converse, let us assume the equality

$$\int_M (|J(h)| - 1) \frac{J(h)}{|J(h)|} (\operatorname{div} Y) \circ h J(h)\, \omega_M = 0. \tag{4}$$

We will show that $J(h)$ is constant.

Consider the function $g := (|J(h)| - 1)\frac{J(h)}{|J(h)|}$ on M and note that $f := g \circ h^{-1}$ is a function on N and $f \circ h = g$. Hence, the integral in display (4) is also given by

$$\int_M f \circ h \, (\operatorname{div} Y) \circ h \, J(h) \, \omega_M.$$

By the change of variables formula,

$$\begin{aligned} \int_M f \circ h \, (\operatorname{div} Y) \circ h \, J(h) \, \omega_M &= \int_{h(M)} f (\operatorname{div} Y) \omega_N \\ &= \int_N f L_Y \omega_N \\ &= \int_N f d i_Y \omega_N. \end{aligned}$$

Because the exterior derivative is a $\wedge$-antiderivation, we have that

$$d(f i_Y \omega_N) = df \wedge i_Y \omega_N + f d i_Y \omega_N;$$

hence,

$$\int_N f d i_Y \omega_N = \int_N df \wedge i_Y \omega_N + \int_N d(f i_Y \omega_N).$$

Since N has no boundary, Stokes's theorem can be used to show that the last integral vanishes.

Thus, to complete the proof, it suffices to show that if

$$\int_N df \wedge i_Y \omega_N = 0$$

for all $Y \in \mathfrak{X}(N)$, then $df = 0$.

Because the interior product is a $\wedge$-antiderivation, we have the identity

$$i_Y(df \wedge \omega_N) = i_Y df \wedge \omega_N - df \wedge i_Y \omega_N.$$

The $(n+1)$-form $df \wedge \omega_N$ vanishes on N. Hence,

$$\int_N df \wedge i_Y \omega_N = \int_N df(Y) \omega_N$$

and, by our hypothesis,

$$\int_N df(Y) \omega_N = 0 \tag{5}$$

for all $Y \in \mathfrak{X}(N)$.

We will show that equality (5) implies $df = 0$. Suppose, on the contrary, that there exists a continuous vector field $Y \in \mathfrak{X}(N)$ such that $df(Y)(n) \neq 0$ for some point $n \in N$. Without loss of generality, we assume the inequality $df(Y)(n) > 0$. Since h is twice continuously differentiable and h^{-1} is continuous, the function

$$f = \left[\Big(|J(h)| - 1 \Big) \frac{J(h)}{|J(h)|} \right] \circ h^{-1}$$

is continuously differentiable. We conclude that the map $df(Y) : N \to \mathbb{R}$ is continuous. Therefore, there exists an open neighborhood $U \subset N$ of the point $n \in N$ so that $df(Y)(p) > 0$ for every $p \in U$.

Let U_1 be an open nonempty subset of N such that its closure is contained in U. Let $\xi : N \to \mathbb{R}$ be a positive bump function such that $\xi(p) = 1$ for $p \in U_1$ and ξ vanishes outside U. Define a new vector field $Z = \xi Y \in \mathfrak{X}(N)$. Note that $df(Z)(p) = \xi(p)\, df(Y)(p) > 0$ for all $p \in U$ and $df(Z)$ vanishes outside U. It follows that

$$\int_N df(Z)\omega_N = \int_U df(Z)\omega_N > 0,$$

in contradiction to equality (5). Hence, $df = 0$. Since N is connected, f is constant. By the definition of f, the Jacobian $J(h)$ is constant. $\square$

Definition 2.4. A function $h \in \mathrm{Diff}^2(M, N)$ is called a *distortion minimal* map if it is a critical point of the total distortion functional Φ; that is, h satisfies the relation (3).

Theorem 2.5. *A function $h \in \mathrm{Diff}^2(M, N)$ is a distortion minimal map if and only if $J(h)$ is the constant function with value $\mathrm{Vol}(N)/\mathrm{Vol}(M)$.*

Proof. As an immediate corollary of proposition 2.3, h is a distortion minimal map if and only if $J(h)$ is constant. If $J(h)$ is constant, with $J(h)(m) = j$ for all $m \in M$, then

$$\int_M h^*\omega_N = j \int_M \omega_M$$

and $j = \mathrm{Vol}(N)/\mathrm{Vol}(M)$ as required. $\square$

We will use some elementary properties of distortion minimal maps.

Lemma 2.6. *Compositions and inverses of distortion minimal maps are distortion minimal maps.*

Proof. The result follows from theorem 2.5 and the identities

$$\begin{aligned} J(\omega_S,\omega_N)(k\circ\ell) &= J(\omega_M,\omega_N)(k)\circ l\, J(\omega_S,\omega_M)(\ell),\\ 1 = J(\omega_M,\omega_M)(k\circ k^{-1}) &= J(\omega_M,\omega_N)(k)\circ k^{-1}\, J(\omega_N,\omega_M)(k^{-1}), \end{aligned}$$

which hold for smooth orientable manifolds (M,ω_M), (N,ω_N), and (S,ω_S) and functions $k\in \mathrm{Diff}(M,N)$ and $\ell\in\mathrm{Diff}(S,M)$. $\square$

Also we will use (the strong form) of Moser's theorem on volume forms, which we state here for the convenience of the reader (see [8]).

Theorem 2.7. *Let τ_t be a family of volume forms defined for $t\in[0,1]$ on a compact manifold M. If*

$$\int_c \tau_t = \int_c \tau_0 \tag{6}$$

for every n-chain c on M, then there exists a one-parameter family of diffeomorphisms $\phi_t : M\to M$ such that

$$\phi_t^*\tau_t = \tau_0 \tag{7}$$

and ϕ_0 is the identity mapping. Moreover, the dependence of $\phi_t(m)$ on $m\in M$ and $t\in[0,1]$ is as smooth as in the family τ_t.

Theorem 2.8. *If (M,ω_M) and (N,ω_N) are diffeomorphic n-dimensional compact connected oriented manifolds without boundary, then (i) there is a distortion minimal map from M to N, (ii) every distortion minimal map from M to N minimizes the strain energy functional Φ, and (iii) the minimum value of the strain energy functional is*

$$\Phi_{min} = \frac{\big(\mathrm{Vol}(M)-\mathrm{Vol}(N)\big)^2}{\mathrm{Vol}(M)}. \tag{8}$$

Proof. To prove (i), choose a diffeomorphism $h\in\mathrm{Diff}^2(M,N)$. The differential form $h^*\omega_N$ is a volume on M. Define a new volume on M as follows:

$$\bar\omega_M = \frac{\mathrm{Vol}(M)}{\int_M h^*\omega_N} h^*\omega_N.$$

Since

$$\int_M \bar\omega_M = \int_M \omega_M$$

and M is compact, by an application of Moser's theorem 2.7, there exists a C^2 diffeomorphism $f : M \to M$ such that $\omega_M = f^*\bar{\omega}_M$. Hence,

$$\frac{\int_M h^*\omega_N}{\operatorname{Vol}(M)}\omega_M = (h \circ f)^*\omega_N;$$

and, since M is connected, we conclude that $J(h \circ f) = \operatorname{Vol}(N)/\operatorname{Vol}(M)$ is constant and $DJ(h \circ f) = 0$. Thus, $k = h \circ f$ is a distortion minimal map.

To prove parts (ii) and (iii), note that if k is an arbitrary distortion minimal map from M to N, then

$$\Phi(k) = \big(|J(k)| - 1\big)^2 \operatorname{Vol}(M) = \frac{\big(\operatorname{Vol}(M) - \operatorname{Vol}(N)\big)^2}{\operatorname{Vol}(M)}.$$

We claim that this value of Φ is its minimum.

Let $g \in \operatorname{Diff}^2(M, N)$. By the Cauchy-Schwartz inequality,

$$\begin{aligned}\Phi(g) &= \int_M \big(|J(g)| - 1\big)^2\omega_M \\ &\geq \frac{1}{\operatorname{Vol}(M)}\Big(\int_M (|J(g)| - 1)\omega_M\Big)^2 \\ &= \frac{1}{\operatorname{Vol}(M)}\big(\operatorname{Vol}(M) - \operatorname{Vol}(N)\big)^2 \\ &= \Phi(k),\end{aligned}$$

as required. □

Example 2.9. Let S_r and S_R be two-dimensional round spheres of radii r and R (respectively) centered at the origin in $\mathbb{R}^3$. Define $h : S_r \to S_R$ by $h(p) = R/r\,p$ for $p = (x, y, z) \in S_r$. We will show that h is a distortion minimal map.

Consider the parametrization of S_r by spherical coordinates

$$\phi_r(u, v) = (r \sin u \cos v, r \sin u \sin v, r \cos u),$$

where $(u, v) \in [0, \pi) \times [0, 2\pi) =: \Omega$, and use the analogous parametrization ϕ_R for S_R.

A basis of the tangent space of S_r at (u, v) is given by the vectors

$$\begin{aligned}X_1 &= (r \cos u \cos v, r \cos u \sin v, -r \sin u), \\ X_2 &= (-r \sin u \sin v, r \sin u \cos v, 0).\end{aligned}$$

The local representation of the standard volume form ω_r on S_r generated by the usual Riemannian metric $g_p(X,Y) = \langle X, Y\rangle$ for $X, Y \in \mathbb{R}^3$, is

$$\phi_r^*\omega_r = \big(\langle X_1, X_1\rangle\langle X_2, X_2\rangle - \langle X_1, X_2\rangle^2\big)^{1/2}\, du \wedge dv = r^2 \sin u\, du \wedge dv.$$

Similarly $\phi_R^*\omega_R = R^2 \sin u\, du \wedge dv$. The pull back of the volume form on N by the diffeomorphism h is

$$h^*\omega_R(p)(Y,Z) = \omega_R\big(h(p)\big)\Big(\frac{R}{r}\,Y, \frac{R}{r}\,Z\Big),$$

where $Y, Z \in T_pS_r$ and $p \in S_r$. Hence, for $(u,v) \in \Omega$ and $\xi, \eta \in T_{(u,v)}\Omega$ such that $\phi_r(u,v) = p$, $D\phi_r(u,v)\xi = Y$ and $D\phi_r(u,v)\eta = Z$, we have that

$$\begin{aligned}
\phi_r^*h^*\omega_R(u,v)(\xi,\eta) &= \omega_R\big(\frac{R}{r}p\big)\Big(\frac{R}{r}Y, \frac{R}{r}Z\Big)\\
&= \omega_R\big(\phi_R(u,v)\big)\big(D\phi_R(u,v)\xi, D\phi_R(u,v)\eta\big)\\
&= R^2 \sin u\, du \wedge dv(\xi,\eta)\\
&= \frac{R^2}{r^2}\phi_r^*\omega_r(u,v)(\xi,\eta).
\end{aligned}$$

These equalities imply that the Jacobian

$$J(\omega_r, \omega_R)(h)(m) = R^2/r^2 = \mathrm{Vol}(S_R)/\mathrm{Vol}(S_r)$$

for all $m \in S_r$; hence, by theorem 2.8, h is a distortion minimal map.

The total distortion of h, by a direct computation, is given by

$$\begin{aligned}
\Phi(h) &= \int\limits_{S_r} (\big|J(h)\big| - 1)^2\omega_r\\
&= \big(\frac{R^2}{r^2} - 1\big)^2\,\mathrm{Vol}(S_r)\\
&= \frac{\big(\mathrm{Vol}(S_R) - \mathrm{Vol}(S_r)\big)^2}{\mathrm{Vol}(S_r)},
\end{aligned}$$

in agreement with theorem 2.8.

Remark 2.10 (Harmonic maps)*.* For $h \in \mathrm{Diff}^2(M,N)$, the distortion functional (2) has value

$$\Phi(h) = \int_M |J(h)|^2\omega_M - 2\,\mathrm{Vol}(N) + \mathrm{Vol}(M).$$

Thus, it suffices to consider the minimization problem for the reduced functional Ψ given by

$$\Psi(h) = \int_M |J(h)|^2 \omega_M.$$

We note that if M and N are one-dimensional, then Ψ is the same as

$$\Psi(h) = \int_M |Dh|^2 \omega_M.$$

An extremal of this functional is called a harmonic map (see [2, 3, 4]). Thus, for the one-dimensional case, distortion minimal maps and harmonic maps coincide. On the other hand, there seems to be no obvious relationship in the general case.

3 Morphs of embedded manifolds

We will discuss a minimization problem for morphs of compact connected boundaryless oriented n-dimensional smooth manifolds embedded in $\mathbb{R}^{n+1}$.

3.1 Pairwise minimal morphs

Definition 3.1. Let M and N be compact connected oriented n-dimensional smooth manifolds without boundary embedded in $\mathbb{R}^{n+1}$. A C^1 function $H : [0,1] \times M \to \mathbb{R}^{n+1}$ is a morph from M to N if the following conditions hold:

(i) $p \mapsto H(t,p)$ is a diffeomorphism on its image for each $t \in I = [0,1]$;

(ii) the image $M^t = H(t,M)$ is an n-dimensional manifold possessing all the properties of M and N mentioned above;

(iii) $p \mapsto H(0,p)$ is a diffeomorphism of M;

(iv) the image of the map $p \mapsto H(1,p)$ is N.

For simplicity, we will only consider morphs H such that $p \mapsto H(0,p)$ is the identity map. We assume that each manifold $M^t = H(t,M)$ (with $M^0 = M$ and $M^1 = N$) is equipped with the volume form $\omega_t = i_{\eta_t}\Omega$, where

$$\Omega = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{n+1}$$

is the standard volume form on $\mathbb{R}^{n+1}$ and $\eta_t : M^t \to \mathbb{R}^{n+1}$ is the outer unit normal vector field on M^t with respect to the usual metric on $\mathbb{R}^{n+1}$. Also, as a convenient notation, we use $h^t = H(t,\cdot) : M \to M^t$.

Definition 3.2. A morph H is *distortion pairwise minimal* (or, for brevity, *pairwise minimal*) if $h^{s,t} = h^t \circ (h^s)^{-1} : M^s \to M^t$ is a distortion minimal map for every s and t.

By proposition 2.3 and theorem 2.8, a morph H is pairwise minimal if and only if each Jacobian $J(\omega_s, \omega_t)(h^{s,t})$ is constant.

Proposition 3.3. *Let $M = M^0$ and $N = M^1$ be n-dimensional manifolds as in definition 3.1 equipped with the (respective) volume forms ω_0 and ω_1. A morph H between M and N is distortion pairwise minimal if and only if*

$$\frac{J(\omega_0, \omega_t)(h^t)(m)}{\mathrm{Vol}(M^t)} = \frac{1}{\mathrm{Vol}(M)} \tag{9}$$

for all $t \in [0,1]$ and $m \in M$.

Proof. Since compositions and inverses of distortion minimal maps are distortion minimal maps and since every distortion minimal map is minimal (according to lemma 2.6 and theorem 2.8), it suffices to prove that each map $h^t : M \to M^t$ is minimal if and only if the map (9) is constant. By theorem 2.5, we conclude that $h^t : M \to M^t$ is a minimal map if and only if

$$J(\omega_0, \omega_t)(h^t)(m) = \frac{\mathrm{Vol}(M^t)}{\mathrm{Vol}(M)}$$

for all $m \in M$, as required. $\square$

Proposition 3.4. *Let M and N be n-dimensional manifolds as in proposition 3.3. If there is a morph G from M to N, then there is a distortion pairwise minimal morph between M and N.*

Proof. Fix a morph G from M to N with the corresponding family of diffeomorphisms $g^t := G(t, \cdot)$, let $M^t := G(t, M)$, and consider the family of volume forms

$$\bar{\omega}_t = \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(M^t)} (g^t)^* \omega_t$$

defined for $t \in [0,1]$. It is easy to see that

$$\int_M \bar{\omega}_t = \int_M \bar{\omega_0};$$

hence, by Moser's theorem 2.7, there is a family of diffeomorphisms α^t on M such that α^t depends continuously on t and $\omega_M = \alpha^t \bar{\omega}_t$. It follows that

$$(g^t \circ \alpha^t)^* \omega_t = \frac{\mathrm{Vol}(M^t)}{\mathrm{Vol}(M)} \omega_M;$$

therefore,

$$J(\omega_M, \omega_t)(g^t \circ \alpha^t)(m) = \frac{\mathrm{Vol}(M^t)}{\mathrm{Vol}(M)}$$

for all $m \in M$. The morph H corresponding to the family $h^t := g^t \circ \alpha^t$ is the desired distortion pairwise minimal morph. $\square$

3.2 Minimal morphs

We will define distortion minimal morphs between embedded connected oriented n-dimensional smooth manifolds without boundary.

For a morph H from M to N, let $E_{s,t}$ denote the total distortion of $h^{s,t} : M^s \to M^t$. We have that

$$\begin{aligned} E_{s,t} &= \int_{M^s} \Big(\big|J(h^{s,t})\big| - 1\Big)^2 \omega_s \\ &= \int_{h^s(M)} \Big(\big|J(h^t) \circ (h^s)^{-1} J((h^s)^{-1})\big| - 1\Big)^2 \omega_s \\ &= \int_M \Big(\big|J(h^t) J((h^s)^{-1}) \circ h^s\big| - 1\Big)^2 J(h^s)\omega_M. \end{aligned}$$

Because $1 = J\big((h^s)^{-1} \circ h^s\big) = J\big((h^s)^{-1}\big) \circ h^s J(h^s)$, it follows that

$$J\big((h^s)^{-1}\big) \circ h^s = 1/J(h^s);$$

and, therefore,

$$E_{s,t} = \int_M \Big(\frac{J(h^t)}{J(h^s)} - 1\Big)^2 J(h^s)\omega_M. \tag{10}$$

If H is a C^2 morph, then $E_{s,t}$ is twice continuously differentiable with respect to s. By Taylor's theorem, $E_{s,t}$ has the representation

$$E_{s,t} = E_{t,t} + \frac{d}{ds}(E_{s,t})\big|_{s=t}(s-t) + \frac{1}{2}\frac{d^2}{ds^2}(E_{s,t})\big|_{s=t}(s-t)^2 + O\big((s-t)^3\big).$$

Note that $E_{t,t}$ and $\frac{d}{ds}(E_{s,t})\big|_{s=t}$ both vanish, and

$$\frac{1}{2}\frac{d^2}{ds^2}(E_{s,t})\big|_{s=t} = \int_M \frac{\Big(\frac{d}{dt}J(h^t)\Big)^2}{J(h^t)}\omega_M.$$

Definition 3.5. The *infinitesimal distortion* of a C^2 morph H from M to N at $t \in [0,1]$ is

$$\varepsilon^H(t) = \lim_{s \to t} \frac{E_{s,t}}{(s-t)^2} = \int_M \frac{\left(\frac{d}{dt} J(h^t)\right)^2}{J(h^t)} \omega_M.$$

The *total distortion functional* Φ defined on such morphs is given by

$$\Phi(H) = \int_0^1 \varepsilon^H(t)dt = \int_0^1 \left(\int_M \frac{\left(\frac{d}{dt} J(h^t)\right)^2}{J(h^t)} \omega_M \right) dt. \tag{11}$$

Definition 3.6. A C^2 morph is called a *distortion minimal extremal* if it is an extremal of the functional Φ with respect to C^2 morphs. A C^2 morph is called a *minimal distortion morph* if it minimizes the functional Φ.

Note that $\varepsilon^H(t)$ depends continuously on t provided that H is a C^2 morph; because, in this case, the integrand is a uniformly continuous function on the compact set $[0,1] \times M$.

Lemma 3.7. *Every distortion minimal C^2 morph is pairwise minimal.*

Proof. Suppose, on the contrary, that a C^2 morph H minimizes Φ but H is not pairwise minimal. By proposition 3.4, there is a pairwise minimal morph G with the same set of intermediate manifolds $M^t = H(t,M) = G(t,M)$. Define the corresponding families of diffeomorphisms for G and H by $g^t(\cdot) = G(t,\cdot)$ and $h^t(\cdot) = H(t,\cdot)$.

Since G is pairwise minimal and H is not, for at least one $t_0 \in (0,1]$ the function $h^{t_0} : M \to M^{t_0}$ is not a distortion minimal diffeomorphism. The diffeomorphism $h^{s,t_0} = h^{t_0} \circ (h^s)^{-1}$, where $s \in [0,1]$ and $s \neq t$ is not distortion minimal. If it were, lemma 2.6 would imply that $h^{t_0} = h^{s,t_0} \circ h^s$ is distortion minimal, in contradiction.

By the definition of E_{s,t_0}, we have that $E_{s,t_0}(H) > E_{s,t_0}(G)$ whenever $s \in [0,1]$ and $s \neq t$. Hence, by the monotonicity property of the limit,

$$\varepsilon^H(t_0) > \varepsilon^G(t_0).$$

Since the function $\varepsilon^H(t) : [0,1] \to \mathbb{R}_+$ is continuous, there exists a neighborhood U of t_0 so that $\varepsilon^H(t) - \varepsilon^G(t) > 0$ for every $t \in U$. Also, since the morph G is pairwise minimal, $\varepsilon^H(t) - \varepsilon^G(t) \geq 0$ for all $t \in [0,1]$. It follows that

$$\Phi(H) - \Phi(G) = \int\limits_0^1 \left(\varepsilon^H(t) - \varepsilon^G(t)\right) dt \geq \int\limits_U \left(\varepsilon^H(t) - \varepsilon^G(t)\right) dt > 0.$$

The latter inequality shows that H is not a minimal morph, in contradiction. □

Corollary 3.8. *The total distortion of a C^2 pairwise minimal morph H from M to N is*

$$\Phi(H) = \int_0^1 \frac{\left(\frac{d}{dt}\operatorname{Vol}(M^t)\right)^2}{\operatorname{Vol}(M^t)}\,dt. \tag{12}$$

Proof. Since H is pairwise minimal, each diffeomorphism $h^t : M \to M^t$ is distortion minimal; hence, $J(\omega_M, \omega_t)(h^t) = \frac{\operatorname{Vol}(M^t)}{\operatorname{Vol}(M)}$ by proposition 3.3. By substitution of this value into formula (11), we obtain the desired value (12). □

Lemma 3.9. *Suppose that H is a C^2 pairwise minimal morph between manifolds M and N, and $M^t = H(t, M)$ for $t \in [0,1]$.*

(i) If all manifolds M^t have the same volume, then H is a minimal morph and its total distortion is equal to zero.

(ii) If $\operatorname{Vol}(M) \neq \operatorname{Vol}(N)$ and $\operatorname{Vol}(M^t)$ changes according to the law

$$\operatorname{Vol}(M^t) = \Big[(\sqrt{\operatorname{Vol}(M)} - \sqrt{\operatorname{Vol}(N)}\,)t - \sqrt{\operatorname{Vol}(M)}\,\Big]^2, \tag{13}$$

then H is a minimal morph with total distortion

$$\Phi_{min} = 4\big(\sqrt{\operatorname{Vol}(N)} - \sqrt{\operatorname{Vol}(M)}\big)^2. \tag{14}$$

Proof. The statement of the lemma follows immediately from an analysis of the minimization problem for the auxiliary functional

$$\Psi(\phi) = \int_0^1 \frac{\dot{\phi}^2}{\phi}\,dt \tag{15}$$

over the class of all positive continuously differentiable functions $\phi : [0,1] \to \mathbf{R}_+$ satisfying $\phi(0) = \operatorname{Vol}(M)$ and $\phi(1) = \operatorname{Vol}(N)$. The latter functional attains its minimum at $\phi(t) \equiv 0$ if $\operatorname{Vol}(M) = \operatorname{Vol}(N)$ and at $\phi(t) = \Big[(\sqrt{\operatorname{Vol}(M)} - \sqrt{\operatorname{Vol}(N)}\,)t - \sqrt{\operatorname{Vol}(M)}\,\Big]^2$ whenever the volumes of M and N are distinct. The lemma follows from this result and formula (12). □

The next result provides a basic class of distortion minimal morphs.

Proposition 3.10. *Suppose that M is an n-dimensional manifold embedded in $\mathbb{R}^{n+1}$ that satisfies the assumptions of definition 3.1. If α is a positive real number and*

$$N := \{\alpha m : m \in M\},$$

then (i) N is a manifold satisfying all of the assumptions of definition 3.1. (ii) the morph given by the family of maps $h^t(m) = \lambda(t)m$, where

$$\lambda(t) = \operatorname{Vol}(M)^{-\frac{1}{n}} \Big[(\sqrt{\operatorname{Vol}(M)} - \sqrt{\operatorname{Vol}(N)}\,)t - \sqrt{\operatorname{Vol}(M)} \Big]^{\frac{2}{n}},$$

is a C^∞ distortion minimal morph between M and N.

Proof. Define $h^t(m) = \lambda(t)m$.

We will first compute the value of the Jacobian $J(h^t) := J(\omega_M, \omega_t)(h^t)$. Let $\xi_1, \dots, \xi_n$ be a basis of $T_m M$ and note that

$$\begin{aligned}
(h^t)^* \omega_M(\xi_1, \dots, \xi_n) &= \omega_t\big((h^t)_* \xi_1, \dots, (h^t)_* \xi_n\big) \\
&= \omega_t(\lambda(t)\xi_1, \dots, \lambda(t)\xi_n) \\
&= \big(\lambda(t)\big)^n \omega_t(\xi_1, \dots, \xi_n).
\end{aligned}$$

Thus, we have that $J(h^t) = \big[\lambda(t)\big]^n$. Since $J(h^t)$ is constant on M, the family h^t defines a pairwise minimal morph H.

Lemma 3.9 applies to impose the required additional condition on $\operatorname{Vol}(M^t)$ in order to check that the morph H is distortion minimal. Indeed, the condition

$$\operatorname{Vol}(M^t) = \big[\lambda(t)\big]^n \operatorname{Vol}(M) = \Big[(\sqrt{\operatorname{Vol}(M)} - \sqrt{\operatorname{Vol}(N)}\,)t - \sqrt{\operatorname{Vol}(M)} \Big]^2,$$

yields

$$\lambda(t) = \operatorname{Vol}(M)^{-\frac{1}{n}} \Big[(\sqrt{\operatorname{Vol}(M)} - \sqrt{\operatorname{Vol}(N)}\,)t - \sqrt{\operatorname{Vol}(M)} \Big]^{\frac{2}{n}}.$$

Note that $\lambda(0) = 1$; and, because

$$\operatorname{Vol}(N) = \int_M (h^1)^* \omega_M = \alpha^n \int_M \omega_M = \alpha^n \operatorname{Vol}(M),$$

we also have

$$\lambda(1) = \Big(\frac{\operatorname{Vol}(N)}{\operatorname{Vol}(M)} \Big)^{\frac{1}{n}} = \alpha,$$

as required.

If follows that the corresponding morph $H(t, m) = \lambda(t)m$ is a C^∞ distortion minimal morph between M and N. $\square$

We now have all the tools to state and prove our main result.

Theorem 3.11. *Let M and N be two n-dimensional manifolds satisfying the assumptions of definition 3.1. If M and N are connected by a C^2 morph, then there exists a distortion minimal morph between M and N.*

Proof. Let G be a morph between M and N. Without loss of generality, we assume that G is pairwise minimal (see proposition 3.4). Set

$$H(t,m) = \lambda(t)G(t,m),$$

where $\lambda : [0,1] \to \mathbb{R}$ is to be determined.

Note that if $M^t = H(t,M)$ and $W^t = G(t,M)$, then

$$\mathrm{Vol}(M^t) = \int_M (h^t)^* \omega_M = \big[\lambda(t)\big]^n \int_M (g^t)^* \omega_M = \big[\lambda(t)\big]^n \, \mathrm{Vol}(W^t).$$

Let

$$\rho(t) := \Big[(\sqrt{\mathrm{Vol}(M)} - \sqrt{\mathrm{Vol}(N)}\,)t - \sqrt{\mathrm{Vol}(M)}\, \Big]^2,$$

and define

$$\lambda(t) = \Big[\frac{\rho(t)}{\mathrm{Vol}(W^t)} \Big]^{\frac{1}{n}}.$$

The corresponding volume $\mathrm{Vol}(M^t) = \rho(t)$ satisfies requirement (ii) of lemma 3.9; therefore, H is distortion minimal if it is pairwise minimal.

To see that H is pairwise minimal, we note that

$$J(h^t)\omega_M = (h^t)^* \omega_M = \big[\lambda(t)\big]^n (g^t)^* \omega_M = \big[\lambda(t)\big]^n J(g^t)\omega_M.$$

Thus, $J(h^t) = \big[\lambda(t)\big]^n J(g^t)$; and therefore

$$\frac{J(h^t)}{\mathrm{Vol}(M^t)} = \frac{\big[\lambda(t)\big]^n J(g^t)}{\rho(t)} = \frac{J(g^t)}{\mathrm{Vol}(W^t)}.$$

By proposition 3.3, the latter fraction is constant because G is pairwise minimal. Hence, H is distortion pairwise minimal by a second application of the same proposition. □

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