PERIODIC SOLUTIONS OF A SYSTEM OF COUPLED OSCILLATORS NEAR RESONANCE *

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Abstract. A system of autonomous ordinary differential equations depending on a small parameter is considered such that the unperturbed system has an invariant manifold of periodic solutions that is not normally hyperbolic but is normally nondegenerate. The bifurcation function whose zeros are the bifurcation points for families of perturbed periodic solutions is determined. This result is applied to find the periodic solutions near resonance for a two degree of freedom mechanical system modeling a rotor interacting with an elastic support.

Key words. coupled oscillator, resonance, normal nondegeneracy.

1. Introduction. In this paper we describe an application of the results in [6] to the bifurcation of periodic solutions in a smooth system of coupled oscillators E_ϵ given by

$$\dot{x}_1 = f_1(x_1) + \epsilon g_1(x_1, \dot{x}_1, x_2, \dot{x}_2),$$
$$\dot{x}_2 = f_2(x_2) + \epsilon g_2(x_1, \dot{x}_1, x_2, \dot{x}_2)$$

where $x_i \in \mathbb{R}^2$, $i = 1, 2$ and $\epsilon \in \mathbb{R}$ when the unperturbed system E_0 satisfies the following conditions:

1. The plane autonomous system $\dot{x}_1 = f_1(x_1)$ has an invariant annulus A consisting of periodic solutions (a period annulus) and every periodic solution in A has the same period, $\eta_1 > 0$. Such a period annulus is called isochronous with period η.
2. The plane autonomous system $\dot{x}_2 = f_2(x_2)$ has a periodic trajectory Γ with period $\eta_2 > 0$ such that either Γ is a hyperbolic limit cycle or Γ belongs to a period annulus and the derivative of an associated period function at Γ does not vanish.
3. There are relatively prime positive integers K_1 and K_2 such that $K_1\eta_1 = K_2\eta_2$. In this case we say the periodic trajectory Γ is in resonance with the period annulus A.

A few comments are in order on the conditions just stated. The prime example of an isochronous period annulus is a period annulus of a linear system. However, given any period annulus and any Poincaré section at a point in the period annulus, there is an associated period function that assigns to each point on the section the time of first return to the section. It is easy to see the requirement of a nonzero derivative of a period function as in (2) above is independent of the choice of section and the point chosen on the periodic trajectory. The hypotheses ensure that $A \times \Gamma$ is an invariant submanifold of the state space for the unperturbed system E_0 of a special type we call a normally nondegenerate period manifold. The condition of normal nondegeneracy defined precisely in §2 ensures the first order bifurcation theory in [6] can be applied and the existence of periodic solutions for the perturbed system near the period manifold can be generically determined by computing the simple zeros of a certain bifurcation function also defined in §2. Of particular interest here is the fact that the

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period manifold for E_0 is not normally hyperbolic. Thus, while the period manifold usually does not persist after perturbation, some of the periodic solutions on the period manifold can persist. The bifurcation function determines the number and the position of these persistent periodic solutions. In this way entrainment phenomena can be studied for perturbations of systems which do not already contain stable periodic solutions. For background material on bifurcation problems of this type in addition to [5], [6] the following references and their bibliographies are suggested [1], [2], [3], [7], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19].

While higher dimensional systems can be studied by the same methods, the four dimensional system E_ϵ illustrates the important features of the general theory and is sufficiently general to have many interesting specializations to physical applications. In §3 we apply the theory to an ubiquitous system of differential equations which we interpret, as in [16], as a model for a rotor interacting with an elastic support. We show the existence of a normally nondegenerate period manifold in case the unperturbed system is weakly nonlinear and also in the fully nonlinear case which corresponds to the rotor strongly influenced by a gravitational field. In both cases the bifurcation function is computed explicitly and the existence of periodic solutions relative to the choice of parameters is determined. These results are augmented by some numerical evidence suggesting the role of these bifurcating families of periodic solutions in determining the global behavior of the perturbed system.

The plan of the paper is as follows. In §2 we review the general theory of [6]. In §3 we specialize the general theory to the case represented by E_ϵ and identify the bifurcation function. These results are applied in §4 to the mechanical system modeling the rotor with elastic support. There the bifurcation function is computed explicitly in terms of elliptic functions and its zeros are computed. This determines the perturbed periodic solutions of the coupled mechanical oscillators near resonance. In addition, §4 contains a discussion of some numerical experiments that suggest the coexistence, for certain choices of the parameters, of perturbed periodic attractors, as predicted by the bifurcation analysis, and more complicated nonperiodic attractors.

2. Bifurcation Theory. In this section we outline for completeness a result in [6] which will be used in the analysis of the system E_ϵ defined in the introduction. The analysis begins with a smooth system of differential equations F_ϵ given by

$$\dot{x} = f(x) + \epsilon g(x, \dot{x}, \epsilon), \quad x \in \mathbb{R}^{n+1}, \quad \epsilon \in \mathbb{R}$$

where the unperturbed system F_0 contains a normally nondegenerate period manifold. Here, a period manifold $\mathcal{A}$ is a smooth invariant connected $(k+1)$-dimensional submanifold of $\mathbb{R}^{n+1}$ consisting entirely of periodic solutions of the unperturbed system with the additional property that the Poincaré map P associated with any Poincaré section Σ is the identity on $\mathcal{A} \cap \Sigma$. Of course, period manifolds generalize to many dimensions the concept of a period annulus. To define the concept of normal nondegeneracy we need a few more definitions. Restricting to a particular Poincaré section Σ_0 which has nonempty intersection with $\mathcal{A}$, there is some $\epsilon_0 > 0$ and some subsection $\Sigma \subseteq \Sigma_0$ such that the parametrized Poincaré map $P : \Sigma \times (-\epsilon_0, \epsilon_0) \to \Sigma_0$ given by $(\xi, \epsilon) \mapsto P(\xi, \epsilon)$ where $P(\xi, \epsilon)$ denotes the first return to Σ_0 of the perturbed solution starting at $\xi \in \Sigma$. After choosing coordinates on Σ, given by $s : \mathbb{R}^n \to \Sigma$, the parameterized Poincaré map is identified with its local representation $p : \mathbb{R}^n \times (-\epsilon_0, \epsilon_0) \to \mathbb{R}^n$ given by $p(y, \epsilon) := s^{-1} P(s(y), \epsilon)$. This, in turn, allows us to define the parametrized displacement function $\delta : \mathbb{R}^n \times (-\epsilon_0, \epsilon_0) \to \mathbb{R}^n$ by $\delta(y, \epsilon) := p(y, \epsilon) - y$. Now, for $y_* \in \mathbb{R}^n$ such that $s(y_*) \in \Sigma \cap \mathcal{A}$, it is clear that the derivative of the map

$y \mapsto \delta(y, 0)$, which we denote by $D\delta(y, 0)$, when evaluated at y_* will have a nontrivial kernel containing the tangent space of $\mathcal{A}$. More precisely, if $v \in \mathbb{R}^n$ and $Ds(y_*)v \in T_{s(y_*)}\Sigma \cap T_{s(y_*)}\mathcal{A}$, then $D\delta(y_*, 0)v = 0$. Since $\Sigma \cap \mathcal{A}$ is k-dimensional, the kernel of $D\delta(y_*, 0)$ has dimension at least k. If this kernel has dimension k for each y such that $s(y) \in \mathcal{A}$, we say $\mathcal{A}$ is normally nondegenerate. Perhaps a remark is in order on the definition of displacement. One must exercise caution when defining displacement on the manifold Σ. We have avoided the differential geometry necessary to give an intrinsic definition by introducing local coordinates. However, it should be clear that the zero set of the displacement function, the set corresponding to periodic solutions of F_ϵ, is invariant under change of coordinates.

A goal of the theory in [6] is the identification of a bifurcation function $\mathcal{B}$ defined on $\Sigma \cap \mathcal{A}$ whose simple zeros correspond to the initial values of persistent periodic solutions of the unperturbed system. To construct the bifurcation function, we start with a splitting of the tangent bundle over $\mathbb{R}^{n+1}$ into three subbundles, $\mathcal{E}$ generated by the unperturbed vector field, $\mathcal{E}^{\text{tan}}$ tangent to $\mathcal{A}$ but complementary to $\mathcal{E}$, and $\mathcal{E}^{\text{nor}}$ normal to $\mathcal{A}$. In particular, for $y \in \mathcal{A}$ we have $\mathbb{R}^{n+1} = \mathcal{E}(y) \oplus \mathcal{E}^{\text{tan}}(y) \oplus \mathcal{E}^{\text{nor}}(y)$. Such a splitting always exists, but the last two summands are not unique. Next, we define special coordinates on $\mathbb{R}^{n+1}$ near each point $y \in \Sigma \times \mathcal{A}$ which respect the splitting. For this, we choose $\Delta : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n+1}$ given by $(s, \theta, \zeta) \mapsto \Delta(s, \theta, \zeta)$ such that (using subscripted variables to denote partial derivatives)

$$\begin{aligned}
\Delta_s(0, \theta, 0) &: \mathbb{R} \to \mathcal{E}(\Delta(0, \theta, 0)),\\
\Delta_\theta(0, \theta, 0) &: \mathbb{R}^k \to \mathcal{E}^{\text{tan}}(\Delta(0, \theta, 0)),\\
\Delta_\zeta(0, \theta, 0) &: \mathbb{R}^{n-k} \to \mathcal{E}^{\text{nor}}(\Delta(0, \theta, 0)).
\end{aligned}$$

Such coordinates are called adapted to the splitting over $\mathcal{A}$. An associated Poincaré section, again denoted by Σ, is given by the image of the map $(\theta, \zeta) \mapsto \Delta(0, \theta, \zeta)$. In these coordinates the kernel of $D\delta(\Delta(0, \theta, 0), 0)$ corresponds to $\mathcal{E}^{\text{tan}}(\Delta(0, \theta, 0))$ and there is a k-dimensional complement to the range of this derivative in $\mathbb{R}^{n+1}$. After choosing coordinates on the range, the linear projection $H(\theta)$ from the tangent space of $\mathbb{R}^{n+1}$ to this range can be represented as a linear map of the form

$$H(\theta) : \mathcal{E} \oplus \mathcal{E}^{\text{tan}} \oplus \mathcal{E}^{\text{nor}}(\Delta(0, \theta, 0)) \to \mathbb{R}^k.$$

Next, let $t \mapsto x(t, \theta)$ denote the solution of F_0 with initial condition $x(0, \theta) = \Delta(0, \theta, 0)$ and consider the variational equation along this solution, namely,

$$\dot{W} = Df(x(t, \theta))W.$$

This variational equation has a fundamental matrix solution $t \mapsto \Phi(t, \theta)$ with initial value $\Phi(0, \theta) = I$. There are parametrized linear maps

$$\begin{aligned}
&a(t, \theta) : \mathcal{E}^{\text{nor}}(x(0, \theta)) \to \mathcal{E}^{\text{tan}}(x(t, \theta)), \quad b(t, \theta) : \mathcal{E}^{\text{nor}}(x(0, \theta)) \to \mathcal{E}^{\text{nor}}(x(t, \theta)),\\
&c(t, \theta) : \mathcal{E}^{\text{tan}}(x(0, \theta)) \to \mathcal{E}^{\text{tan}}(x(t, \theta)), \quad d(t, \theta) : \mathcal{E}^{\text{nor}}(x(0, \theta)) \to \mathcal{E}(x(t, \theta)),\\
&e(t, \theta) : \mathcal{E}^{\text{tan}}(x(0, \theta)) \to \mathcal{E}(x(t, \theta)),
\end{aligned}$$

such that the block form of $\Phi(t, \theta)$ with respect to the splitting is

$$\Phi(t, \theta) = \begin{pmatrix} 1 & e(t, \theta) & d(t, \theta) \\ 0 & c(t, \theta) & a(t, \theta) \\ 0 & 0 & b(t, \theta) \end{pmatrix}$$

and such that

$$e(0,\theta)=0,\;\; d(0,\theta)=0,\;\; c(0,\theta)=I,\;\; a(0,\theta)=0,\;\; b(0,\theta)=I.$$

Also, the vector field along the unperturbed solution defined by the perturbation, namely, $G(t,\theta) := g(x(t,\theta),\dot{x}(t,\theta),0))$, has a representation relative to the splitting given by

$$G(t,\theta)=\begin{pmatrix} G^{\mathcal{E}}(t,\theta) \\ G^{\tan}(t,\theta) \\ G^{\mathrm{nor}}(t,\theta) \end{pmatrix}.$$

Here, $G(t,\theta)$ is the derivative of $f(x)+\epsilon g(x,\dot{x},\epsilon)$ with respect to ϵ evaluated at $\epsilon=0$. The bifurcation function for the system F_ϵ adapted to the period manifold $\mathcal{A}$ is the function $\mathcal{B}:\mathbb{R}^k\to\mathbb{R}^k$ defined by

$$\mathcal{B}(\theta)=H(\theta)\begin{pmatrix} 0 \\ \mathcal{N}(\theta) \\ \mathcal{M}(\theta) \end{pmatrix}$$

where

$$\mathcal{M}(\theta) := \int_0^{T(\theta)} b^{-1}(s,\theta)G^{\mathrm{nor}}(s,\theta)\,ds,$$
$$\mathcal{N}(\theta) := \int_0^{T(\theta)} c^{-1}(s,\theta)G^{\tan}(s,\theta)-c^{-1}(s,\theta)a(s,\theta)b^{-1}(s,\theta)G^{\mathrm{nor}}(s,\theta)\,ds$$

and where $T(\theta)$ denotes the time of first return to the Poincaré section for the unperturbed solution $t\mapsto x(t,\theta)$. The following theorem is proved in [6].

THEOREM 2.1. *Suppose F_ϵ given by*

$$\dot{x}=f(x)+\epsilon g(x,\dot{x},\epsilon),\;\; x\in\mathbb{R}^{n+1},\;\; \epsilon\in\mathbb{R}$$

has a normally nondegenerate period manifold $\mathcal{A}$ with adapted coordinate system given by $(s,\theta,\zeta)\mapsto\Delta(s,\theta,\zeta)$. If θ_0 is a simple zero of the bifurcation function $\theta\to\mathcal{B}(\theta)$ adapted to $\mathcal{A}$, then there is an $\epsilon_*>0$ and a smooth function $\beta:(-\epsilon_*,\epsilon_*)\to\mathbb{R}^k\times\mathbb{R}^{n-k}$ with $\beta(0)=(\theta_0,0)$ such that $\Delta(0,\beta(\epsilon))$ is the initial value for a periodic solution of F_ϵ. In this section we apply the results outlined in §2 to the system E_ϵ defined in the introduction. To do this we must identify the bifurcation function. Other, perhaps simpler examples of the identification procedure are given in [6]. In any case, there are several steps.

Step 1. [Definition of the period manifold] Under the assumptions 1–3 listed in the introduction, the unperturbed system E_0 has a three dimensional period manifold given by $\mathcal{A}:=A\times\Gamma$. In fact, every solution of the unperturbed system starting on $\mathcal{A}$ has the same period $T_{\mathcal{A}}:=K_1\eta_1$.

Step 2. [Adapted coordinates] For vectors $v=(v_1,v_2)$ and $w=(w_1,w_2)$ in $\mathbb{R}^2$, let $<v,w>$ denote the usual inner product, $||v||^2 :=<v,v>$, $v^\perp:=(-v_2,v_1)$ and $v\wedge w:=<w,v^\perp>$. Using these definitions and the unperturbed vector fields f_1 and f_2 on $\mathbb{R}^2$, we define two smooth vector fields $f_1^\perp$ and $f_2^\perp$ on $\mathbb{R}^2$. Also, we let φ^i denote the flow of $\dot{x}_i=f_i(x_i)$ and ψ^i denote the flow of $\dot{x}_i=f_i^\perp(x_i)$ for $i=1,2$. For each

$x = (x_1, x_2)$ in $A \times \Gamma$, we define a splitting over $\mathcal{A}$ by

$$
\begin{aligned}
\mathcal{E}(x) &= \left[\begin{pmatrix} f_1(x_1) \\ f_2(x_2) \end{pmatrix} \right], \\
\mathcal{E}^{\tan}(x) &= \left[\begin{pmatrix} ||f_1(x_1)||^{-2} f_1^{\perp}(x_1) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ f_2(x_2) \end{pmatrix} \right], \\
\mathcal{E}^{\mathrm{nor}}(x) &= \left[\begin{pmatrix} 0 \\ ||f_2(x_2)||^{-2} f_2^{\perp}(x_2) \end{pmatrix} \right]
\end{aligned}
$$

where the square brackets here and hereafter denote the subspace spanned by the enclosed vectors. This gives

$$T_x \mathcal{A} = \mathcal{E}(x) \oplus \mathcal{E}^{\tan}(x) \oplus \mathcal{E}^{\mathrm{nor}}(x).$$

Next, fix $\xi_1 \in A$ and $\xi_2 \in \Gamma$ and define adapted coordinates $\Delta : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$(s, p, q, \zeta) \mapsto \left(\varphi_s^1(\psi_p^1(\xi_1)), \psi_\zeta^2(\varphi_{s+q}^2(\xi_2)) \right).$$

If

$$\Sigma_0 := \left\{ \Delta(0, p, q, \zeta) \mid (p, q, \zeta) \in \mathbb{R}^3 \right\},$$

then there is some open subset $\Sigma \subseteq \Sigma_0$ that is a three dimensional Poincaré section for E_0 at (ξ_1, ξ_2).

Step 3. [Fundamental matrix of variational equation in adapted coordinates] We consider the fundamental matrix solution $\Phi(t)$ with initial condition $\Phi(0) = I$ for the variational equation

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} Df_1\left(\varphi_t^1(\psi_p^1(\xi_1))\right) & 0 \\ 0 & Df_2\left(\varphi_{t+q}^2(\xi_2)\right) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and recall Diliberto's theorem [5, 6, 8].

THEOREM 2.2 (Diliberto's Theorem [5, 8]). *If $\dot{x} = f(x),\ x \in \mathbb{R}^2,\ f(\xi) \neq 0$, and $t \mapsto x(t, p)$ is the solution of the differential equation such that $x(0, p) = p$, then the homogeneous variational equation*

$$\dot{W} = Df(x(t, \xi))W$$

has a fundamental matrix solution $t \mapsto \Psi(t)$

$$\Psi(t) = \begin{pmatrix} 1 & \alpha(t, \xi) \\ 0 & \beta(t, \xi) \end{pmatrix}$$

with respect to the moving frame

$$\left\{ f(t, \xi),\ ||f(t, \xi)||^{-2} f^{\perp}(t, \xi) \right\}$$

where

$$
\begin{aligned}
& f(t, \xi) := f(x(t, \xi)) \\
& \beta(t, \xi) \ = \exp \int_0^t \operatorname{div} f(s, \xi)\, ds, \\
& \alpha(t, \xi) \ = \int_0^t \left\{ \frac{1}{||f||^2} (2\kappa ||f|| - \operatorname{curl} f)\beta \right\} (s, \xi)\, ds
\end{aligned}
$$

and κ denotes the signed scalar curvature

$$\kappa(t,\xi) := \frac{1}{||f(t,\xi)||^3} f(t,\xi) \wedge Df(t,\xi) f(t,\xi).$$

Also, to compress the notation, we define

$$\begin{array}{ll}\alpha_1(s,p) := \alpha_1(s,\psi_p^1(\xi_1)), & \beta_1(s,p) := \beta_1(s,\psi_p^1(\xi_1)),\\ f_1(s,p) := f_1(\varphi_s^1(\psi_p^1(\xi_1))), & f_1^\perp(s,p) := f_1^\perp(\varphi_s^1(\psi_p^1(\xi_1))),\\ \alpha_2(s,q) := \alpha_2(s,\varphi_q^2(\xi_2)), & \beta_2(s,q) := \beta_2(s,\varphi_q^2(\xi_2)),\\ f_2(s,q) := f_2(\varphi_{s+q}^2(\xi_2)), & f_2^\perp(s,p) := f_2^\perp(\varphi_{s+q}^2(\xi_2))\end{array}$$

where the subscripts on α and β refer to the functions as defined in Diliberto's theorem for the unperturbed equations $\dot{x}_i = f_i(x_1)$, $i = 1,2$. Now, the fundamental matrix solution relative to the basis $\mathcal{S}$ for our splitting

$$\left\{F(t,p,q), F_1^{\tan}(t,p), F_2^{\tan}(t,q), F^{\text{nor}}(t,q)\right\} :=$$
$$\left\{\begin{pmatrix} f_1(t,p) \\ f_2(t,q)\end{pmatrix}, \begin{pmatrix} ||f_1(t,p)||^{-2} f_1^\perp(t,p) \\ 0\end{pmatrix}, \begin{pmatrix} 0 \\ f_2(t,q)\end{pmatrix}, \begin{pmatrix} 0 \\ ||f_2(t,q)||^{-2} f_2^\perp(t,q)\end{pmatrix}\right\}$$

is given by

$$\Phi(t) = \begin{pmatrix} 1 & \alpha_1(t,p) & 0 & 0 \\ 0 & \beta_1(t,p) & 0 & 0 \\ 0 & 0 & 1 & \alpha_2(t,q) \\ 0 & 0 & 0 & \beta_2(t,q)\end{pmatrix}.$$

This means the associated maps a, b and c defined in §2 reduce as follows:
$a : \mathcal{E}^{\text{nor}}(\psi_p^1(\xi_1), \varphi_q^2(\xi_2)) \to \mathcal{E}^{\tan}(\varphi_t^1(\psi_p^1(\xi_1)), \varphi_t^1(\varphi_q^2(\xi_2)))$ is given by the 2×1 matrix

$$\begin{pmatrix} 0 \\ \alpha_2(t,q)\end{pmatrix},$$

$b : \mathcal{E}^{\text{nor}}(\psi_p^1(\xi_1), \varphi_q^2(\xi_2)) \to \mathcal{E}^{\text{nor}}(\varphi_t^1(\psi_p^1(\xi_1)), \varphi_t^1(\varphi_q^2(\xi_2)))$ is given by the 1×1 matrix $(\beta_2(t,q))$ and $c : \mathcal{E}^{\tan}(\psi_p^1(\xi_1), \varphi_q^2(\xi_2)) \to \mathcal{E}^{\tan}(\varphi_t^1(\psi_p^1(\xi_1)), \varphi_t^1(\varphi_q^2(\xi_2)))$ is given by the 2×2 matrix

$$\begin{pmatrix} \beta_1(t,p) & 0 \\ 0 & 1\end{pmatrix}.$$

Step 4 [Normal nondegeneracy] Define the transit time map $T : \mathbb{R}^3 \to \mathbb{R}$ given by $(p,q,\zeta) \mapsto T(p,q,\zeta)$ where $T(p,q,\zeta)$ denotes the time of first return of the point $\Delta(0,p,q,\zeta) \in \Sigma$ to Σ_0 and note $T(p,q,0) \equiv T_{\mathcal{A}}$. To show the normal nondegeneracy we must show the kernel of the derivative of the displacement at each point on $\xi \in \Sigma \cap \mathcal{A}$ is two dimensional. In the present case, since we already know the kernel contains the subspace $\mathcal{E}^{\tan}(\xi)$, it suffices to show the derivative of the Poincaré map at ξ is not the identity. To prove this we show

$$DP\left(\psi_p^1(\xi_1), \varphi_q^2(\xi_2), 0\right)\begin{pmatrix} 0 \\ f_2^\perp(0,q)\end{pmatrix} \neq \begin{pmatrix} 0 \\ f_2^\perp(0,q)\end{pmatrix}.$$

The vector in the last formula is tangent to the curve

$$\zeta \mapsto \left(\psi_p^1(\xi_1), \psi_\zeta^2(\varphi_q^2(\xi_2)\right)$$

at $\zeta = 0$. So, we must compute the tangent to the curve

$$\zeta \mapsto P\left(\psi_p^1(\xi_1), \psi_\zeta^2(\varphi_q^2(\xi_2))\right) = \left(\varphi_{T(p,q,0)}^1(\psi_p^1(\xi_1)), \varphi_{T(p,q,0)}^2(\psi_\zeta^2(\varphi_q^2(\xi_2)))\right)$$

at $\zeta = 0$. The computation is just an application of Diliberto's theorem. In fact, we obtain

$$DP\left(\psi_p^1(\xi_1), \varphi_q^2(\xi_2), 0\right)\begin{pmatrix} 0 \\ f_2^\perp(0,q) \end{pmatrix} = \begin{pmatrix} 0 \\ D\varphi_{T(p,q,0)}^2(\varphi_q^2(\xi_2)) f_2^\perp(0,q) \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ ||f_2(0,q)||^2\left(\alpha_2(T(p,q,0),q) f_2(0,q) + \beta_2(T(p,q,0),q)||f_2(0,q)||^{-2} f_2^\perp(0,q)\right) \end{pmatrix}.$$

The infinitesimal displacement of our vector is given by

$$\begin{aligned}\mathcal{R}(q) &:= DP(\psi_p^1(\xi_1), \varphi_q^2(\xi_2), 0)\begin{pmatrix} 0 \\ f_2^\perp(0,0) \end{pmatrix} - \begin{pmatrix} 0 \\ f_2^\perp(0,0) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ ||f_2(0,q)||^2 \alpha_2(T_{\mathcal{A}},q) f_2(0,q) + (\beta_2(T_{\mathcal{A}},q) - 1) f_2^\perp(0,q) \end{pmatrix}.\end{aligned}$$

To see that $\mathcal{R}(q) \neq 0$, we use the following facts: $\beta_2(T_{\mathcal{A}},q)$ is the characteristic multiplier of Γ and the derivative of the transit time function at Γ is given by $-||f_2(0,0)||\alpha_2(T_{\mathcal{A}},q)$, see [5] or [6] for more explanation. Since, by the hypotheses stated in §1, either Γ is hyperbolic or Γ belongs to a period annulus such that the derivative of a period function does not vanish at Γ, it follows that $\mathcal{A}$ is normally nondegenerate.

Step 5. [Projection to Complement of the range of $D\delta(p,q,0,0)$] It is clear from step 4 that a two dimensional complement for the range of $D\delta(p,q,0,0)$, expressed with respect to the basis $\mathcal{S}$ for the splitting over $\mathcal{A}$, is given by

$$\left\{F_1^{\tan}(0,p), \mathcal{R}^\perp(q)\right\}$$

where

$$\mathcal{R}^\perp(q) := \begin{pmatrix} 0 \\ (1 - \beta_2(T(p,q,0),q) f_2(0,q) + ||f_2(0,q)||^2 \alpha_2(T_{\mathcal{A}},q) f_2^\perp(0,q) \end{pmatrix}.$$

Moreover, since

$$\left\{F(0,p,q), F_1^{\tan}(0,p), \mathcal{R}(q), \mathcal{R}^\perp(q)\right\}$$

is a basis $\mathcal{T}$ for $\mathbb{R}^4$, the projection from the original splitting to the chosen complement for the range is easy to compute. In fact, there are four functions, each mapping $\mathbb{R}$ to $\mathbb{R}$, given by $q \mapsto k_1(q)$, $q \mapsto k_2(q)$, $q \mapsto B(q)$ and $q \mapsto C(q)$ such that

$$\begin{aligned} F_2^{\tan}(0,q) &= k_1(q)\mathcal{R}(q) + B(q)\mathcal{R}^\perp(q), \\ F^{\mathrm{nor}}(0,q) &= k_2(q)\mathcal{R}(q) + C(q)\mathcal{R}^\perp(q). \end{aligned}$$

Thus, the matrix of the required projection

$$H(p,q) : \left(\mathcal{E} \oplus \mathcal{E}^{\tan} \oplus \mathcal{E}^{\mathrm{nor}}\right)(\Delta(0,p,q,0)) \to \mathbb{R}^2$$

with respect to the (ordered) basis $\mathcal{S}$ on its domain and the (ordered) basis $\mathcal{T}$ on its range is given by the linear map

$$H(p,q)\begin{pmatrix}\varepsilon\\ \tau_1\\ \tau_2\\ \eta\end{pmatrix}=\begin{pmatrix}\tau_1\\ B(q)\tau_2+C(q)\eta\end{pmatrix}$$

where

$$B(q):=\frac{1-\beta_2(T_{\mathcal{A}},q)}{||f_2(0,q)||^4\alpha_2(T_{\mathcal{A}},\xi_2)^2+(1-\beta_2(T_{\mathcal{A}},q))^2},$$
$$C(q):=\frac{\alpha_2(T_{\mathcal{A}},q)}{||f_2(0,q)||^4\alpha_2(T_{\mathcal{A}},\xi_2)^2+(1-\beta_2(T_{\mathcal{A}},q))^2}.$$

Step 6. [Adapted Components for perturbation] The derivative with respect to ϵ at $\epsilon=0$ of the vector field associated with E_ϵ along the unperturbed solution is given by

$$G(t,p,q):=\begin{pmatrix}g_1(t,p,q)\\ g_2(t,p,q)\end{pmatrix}:=\begin{pmatrix}g_1(x_1(t,p),\dot{x}_1(t,p),x_2(t,q),\dot{x}_2(t,q),0)\\ g_2(x_1(t,p),\dot{x}_1(t,p),x_2(t,q),\dot{x}_2(t,q),0)\end{pmatrix}$$

where $t\mapsto(x_1(t,p),x_2(t,q))$ is the unperturbed solution starting at $\Delta(0,p,q,0)$. The vector $G(t,p,q)$ has a unique expression as a linear combination of the vectors in the basis $\mathcal{S}$. In fact, we suppose

$$G(t,p,q)=\varepsilon F(t,p,q)+\tau_1F_1^{\tan}(t,p)+\tau_2F_2^{\tan}(t,q)+\eta F^{\mathrm{nor}}(t,q)$$

and compute inner products with respect to f_1, $f_1^\perp$, f_2 and $f_2^\perp$ to obtain

$$G^{\tan}(t,p,q):=\begin{pmatrix}\tau_1(t,p,q)\\ \tau_2(t,p,q)\end{pmatrix},$$
$$G^{\mathrm{nor}}(t,p,q):=\eta(t,p,q),$$

where

$$\tau_1(t,p,q)=f_1(t,p)\wedge g_1(t,p,q),$$
$$\tau_2(t,p,q)=\frac{1}{||f_2(t,q)||^2}\langle g_2(t,p,q),f_2(t,q)\rangle-\frac{1}{||f_1(t,p)||^2}\langle g_1(t,p,q),f_1(t,p)\rangle,$$
$$\eta(t,p,q)=f_2(t,q)\wedge g_2(t,p,q).$$

Step 7. [Bifurcation function] Using the definitions of §2 and the results of steps 3–4 we now have

$$\mathcal{M}(p,q)=\int_0^{T_{\mathcal{A}}}b^{-1}(s,q)G^{\mathrm{nor}}(s,p,q)\,ds$$

given by

$$\int_0^{T_{\mathcal{A}}}\frac{1}{\beta_2(s,q)}f_2(t,q)\wedge g_2(t,p,q)\,ds,$$

and

$$\mathcal{N}(p,q) = \int_0^{T_\mathcal{A}} c^{-1}(t,p)G^{\tan}(s,p,q) - c^{-1}(t,p)a(t,q)b^{-1}(t,q)G^{\rm nor}(s,p,q)\,ds$$
$$:= \begin{pmatrix} \mathcal{N}_1(p,q) \\ \mathcal{N}_2(p,q) \end{pmatrix}$$

given by

$$\mathcal{N}_1(p,q) = \int_0^{T_\mathcal{A}} \frac{1}{\beta_1(s,p)} f_1(t,p) \wedge g_1(t,p,q)\,ds,$$
$$\mathcal{N}_2(p,q) = \int_0^{T_\mathcal{A}} \frac{1}{||f_2(t,q)||^2}\langle g_2(t,p,q), f_2(t,q)\rangle - \frac{1}{||f_1(t,p)||^2}\langle g_1(t,p,q), f_1(t,p)\rangle - \frac{\alpha_2(t,q)}{\beta_2(t,q)} f_2(t,q)\wedge g_2(t,p,q)\,ds.$$

Thus, the bifurcation function is given by

$$\mathcal{B}(p,q) = H(p,q)\begin{pmatrix} 0 \\ \mathcal{N}_1(p,q) \\ \mathcal{N}_2(p,q) \\ \mathcal{M}(p,q)\end{pmatrix} = \begin{pmatrix} \mathcal{N}_1(p,q) \\ B(q)\mathcal{N}_2(p,q) + C(q)\mathcal{M}(p,q)\end{pmatrix}.$$

In practice, it is more convenient to clear the nonzero denominator of the second component and to use the normalized bifurcation function given by

$$\mathcal{C}(p,q) := \begin{pmatrix} \mathcal{N}_1(p,q) \\ (1-\beta_2(T_\mathcal{A},q))\,\mathcal{N}_2(p,q) + \alpha_2(T_\mathcal{A},q)\mathcal{M}(p,q)\end{pmatrix}.$$

Of course, $\mathcal{C}$ and $\mathcal{B}$ have the same set of simple zeros.

3. Applications. We consider an application to the model equations for a flywheel attached to an elastic support as described in [16]. The model equations are typical for resonance phenomena and are given by

$$\ddot{z} + \omega^2 z = \frac{\epsilon}{m}\left(-f(z) - \beta\dot{z} + q_1\dot{\theta}^2\cos\theta\right) + O(\epsilon^2),$$
$$\ddot{\theta} = \epsilon\left(\frac{1}{J_0}M_1(\dot{\theta}) + q_2\,\mathrm{g}\sin\theta - q_2\omega^2 z\sin\theta\right) + O(\epsilon^2),$$

where z denotes the displacement of the flywheel relative to its support, θ denotes the angular position of the rotating flywheel relative to the (upward) vertical, g is the gravitational constant and M_1 is the motor characteristic. The remaining parameters are all constant with, of course, ϵ being a small parameter. To apply the results of §3, we write the model equation as a first order system using the transformation $x = \dot{\theta}\cos\theta$, $y = \dot{\theta}\sin\theta$, and assuming $\dot{\theta} > 0$ to obtain

$$\dot{z} = -\omega w,$$
$$\dot{w} = \omega z - \epsilon g(z,w,x,y),$$
$$\dot{x} = -y\sqrt{x^2+y^2} + \epsilon\frac{x}{\sqrt{x^2+y^2}}h(z,w,x,y) + O(\epsilon^2),$$
$$\dot{y} = x\sqrt{x^2+y^2} + \epsilon\frac{y}{\sqrt{x^2+y^2}}h(z,w,x,y) + O(\epsilon^2),$$

where

$$g(z,w,x,y) = \frac{1}{m\omega}\left(-f(z) - \beta w + q_1 x\sqrt{x^2+y^2}\right),$$
$$h(z,w,x,y) = \frac{1}{J_0}M_1\left(\sqrt{x^2+y^2}\right) + q_2\,\mathrm{g}\frac{y}{\sqrt{x^2+y^2}} - q_2\omega^2\frac{yz}{\sqrt{x^2+y^2}}.$$

The transformation to (x,y) variables is geometrically a coordinate chart on the tangent bundle of the circle

$$\{(\theta,\dot\theta)\mid \theta\in\mathbb{S}^1 \text{ and } \dot\theta\in\mathbb{R}\}.$$

The chart does not contain the zero section ($\dot\theta = 0$), but this set is not near the resonance. In fact, the first oscillator is linear with its period annulus A having period $2\pi/\omega$ while the second oscillator has a period annulus at the origin whose period function is given by $r\mapsto 2\pi/r$ where $r := \sqrt{x^2+y^2}$. The primary resonance is given by $r=\omega$. In other words, the resonant periodic solution Γ in the second oscillator lies on the invariant circle of radius ω. With

$$f_1(z,w) := -w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w}, \qquad f_2(x,y) := -y\sqrt{x^2+y^2}\frac{\partial}{\partial x} + x\sqrt{x^2+y^2}\frac{\partial}{\partial y},$$

$\xi_1 := (1,0)$ and $\xi_2 := (\omega,0)$, we find the solution of the unperturbed system with initial value $(\phi^1_p(\xi_1),\varphi^2_q(\xi_2))$ to be given by

$$\begin{aligned} z(t,p) &= \mathrm{e}^{-p}\cos(\omega t), & w(t,p) &= \mathrm{e}^{-p}\sin(\omega t),\\ x(t,q) &= \omega\cos(\omega(t+q)), & y(t,q) &= \omega\sin(\omega(t+q)). \end{aligned}$$

From the results of §3, the bifurcation function is

$$\begin{aligned}\mathcal{B}(p,q) &= \left(\omega\int_0^{2\pi/\omega} wg(z,w,x,y)\,dt,\ -\int_0^{2\pi/\omega}(x^2+y^2)h(z,w,x,y)\,dt\right)\\ &= \left(\frac{\pi\omega}{m}\mathrm{e}^{-2p}(\beta - q_1\omega\,\mathrm{e}^{p}\sin(\omega q)),\ -\frac{\pi\omega^2}{J_0}(2M_1(1/\omega) - J_0q_2\omega^2\,\mathrm{e}^{-p}\sin(\omega q))\right).\end{aligned}$$

The bifurcation function has either 0, 1 or 2 zeros depending on the values of the parameters. The zeros are obtained when the following two equations can be solved for both p and q :

$$\mathrm{e}^{2p} = \frac{\beta\omega J_0 q_2}{2M_1(1/\omega)q_1}, \qquad \sin\omega q = \frac{\beta}{\omega q_1}\sqrt{\frac{2M_1(1/\omega)q_1}{\beta\omega J_0 q_2}}.$$

If we choose $\beta=\omega=J_0=q_1=q_2=1$ and $M_1(1)=1/4$, then these equations reduce to

$$\mathrm{e}^{p} = \sqrt{2}, \qquad \sin q = \frac{1}{\sqrt{2}}.$$

In this case $p=\ln\sqrt{2}$, $q=\pi/4,\,3\pi/4$ are zeros of the bifurcation function. Since the bifurcation function can be normalized to

$$(p,q)\mapsto\left(\beta - q_1\omega\,\mathrm{e}^{p}\sin(\omega q),\ 2M_1(1/\omega) - J_0q_2\omega^2\,\mathrm{e}^{-p}\sin(\omega q)\right),$$

a map whose Jacobian is

$$2q_1q_2J_0\omega^2\sin(\omega q)\cos(\omega q),$$

its zeros are simple except when $\sin(\omega q)=\pm 1$. In particular, the zeros of the numerical example are simple and, by the results of §3, there are two bifurcating families of periodic solutions in the model equations for the flywheel with elastic support. We emphasize that although the analysis uses only the $O(\epsilon)$ terms of the model, our result is valid for small ϵ for the full model equations, compare [16].

The analysis just given is prototypical. However, there are other resonances to consider. Using the notation defined above, the general resonance relation is given by

$$K_1\frac{2\pi}{\omega}=K_2\frac{2\pi}{r}$$

or $r=K_2\omega/K_1$. On the resonant orbit

$$x(t,q)=\frac{K_2}{K_1}\omega\cos\frac{K_2}{K_1}\omega(t+q),\qquad y(t,q)=\frac{K_2}{K_1}\omega\sin\frac{K_2}{K_1}\omega(t+q).$$

Thus, the first component of the bifurcation function is given by

$$\begin{aligned}
&\omega\int_0^{K_1 2\pi/\omega} wg(z,w,x,y)\,dt\\
&=\frac{1}{m}\int_0^{K_1 2\pi/\omega}\mathrm{e}^{-p}\sin\omega t\left(-f(z)-\beta\,\mathrm{e}^{-p}\sin\omega t+q_1\left(\frac{K_2}{K_1}\omega\right)^2\cos\frac{K_2}{K_1}\omega(t+q)\right)\,dt\\
&=\frac{1}{m}\,\mathrm{e}^{-2p}\frac{K_1\pi}{\omega}+\frac{q_1}{m}\left(\frac{K_2}{K_1}\omega\right)^2\mathrm{e}^{-p}I_0(q)
\end{aligned}$$

where

$$I_0(q):=\int_0^{K_1 2\pi/\omega}\sin\omega t\cos\frac{K_2}{K_1}\omega(t+q)\,dt.$$

As $I_0(q)$ is nonzero only when $K_1=K_2$, nondegenerate bifurcation to periodic orbits occurs only for the primary resonance.

Up to this point we have assumed several forces are small. To illustrate the possibility of relaxing this hypothesis consider the rotor to be influenced strongly by a "gravitational" force. It is convenient to measure the inclination of the rotor by the angle of displacement from the direction of the gravitational force, downward vertical, i.e., we use the angle $\psi=-\theta-\pi$. The model equations (up to first order in ϵ) become (to first order)

$$\begin{aligned}
\ddot z+\omega^2 z&=\frac{\epsilon}{m}\left(-f(z)-\beta\dot z+q_1\dot\psi^2(-\cos\psi)\right),\\
\ddot\psi&=-\epsilon\left(\frac{1}{J_0}M_1(-\dot\psi)+q_2\,\mathrm{g}\sin\psi-q_2\omega^2 z\sin\psi\right).
\end{aligned}$$

To study the strong gravitational effect we assume $\mathrm{g}:=G/\epsilon$ and transform the independent variable by $\tau=t\sqrt{q_2G}$ to obtain

$$\begin{aligned}
q_2Gz''+\omega^2 z&=-\frac{\epsilon}{m}\left(f(z)+\beta z'\sqrt{q_2G}+q_1q_2G(\psi')^2\cos\psi\right),\\
q_2G\psi''+q_2G\sin\psi&=-\epsilon\left(\frac{1}{J_0}M_1(-\psi'\sqrt{q_2G})-q_2\omega^2 z\sin\psi\right),
\end{aligned}$$

which we rewrite in the form

$$\ddot{z} + \Omega^2 z = \epsilon g(z, \dot{z}, \theta, \dot{\theta}),$$
$$\ddot{\theta} + \sin\theta = \epsilon h(z, \dot{z}, \theta, \dot{\theta}),$$

where

$$g(z, \dot{z}, \theta, \dot{\theta}) := -\Big(F(z) + \lambda \dot{z} + A\dot{\theta}^2 \cos\theta\Big),$$
$$h(z, \dot{z}, \theta, \dot{\theta}) := -\Big(M(\dot{\theta}) - Bz\sin\theta\Big),$$

and the new parameters and functions have the obvious meaning. In particular,

$$\lambda := \frac{\beta}{m}(q_2 G)^{-1/2}, \qquad \Omega := \omega(q_2 G)^{-1/2}, \qquad A := \frac{q_1}{m}, \qquad B := \frac{\omega^2}{G}.$$

The first oscillator, corresponding to the elastic support, has the entire punctured phase plane as an isochronous period annulus with period $2\pi/\Omega$. In fact, if we view the system in the phase plane as

$$\dot{z} = -\Omega w, \qquad\qquad \dot{w} = \Omega z - \frac{\epsilon}{\Omega} g(z, \dot{z}, \theta, \dot{\theta}),$$

the solution of the unperturbed oscillator with initial value $(\mathrm{e}^{-p}, 0)$ is given by

$$z(t,p) = \mathrm{e}^{-p}\cos\Omega t, \qquad\qquad w(t,p) = \mathrm{e}^{-p}\sin\Omega t.$$

The second oscillator models the rotor influenced by a gravitational field. The unperturbed second oscillator is a mathematical pendulum. It has a period annulus (with strictly monotone period function) surrounding the origin of the phase plane. This period annulus corresponds to the nonrotational oscillations of the pendulum. Also, there is a period annulus in the phase cylinder (with strictly monotone period function) corresponding to the rotational oscillations. Thus we have the hypotheses required to apply the theoretical results of §3. The analysis to follow uses elliptic functions. Perhaps this can be avoided?

To compute the bifurcation function we require the time dependent solutions of the mathematical pendulum given in the phase plane by the first order system

$$\dot{\theta} = v, \qquad\qquad \dot{v} = -\sin\theta.$$

For the convenience of the reader and to fix notation, we will outline the usual derivation.

Consider the period annulus in the phase plane. The mathematical pendulum has the first integral $I := v^2/2 - \cos\theta$. For a periodic trajectory Γ let $(a, 0)$ denote the coordinates of its intersection with the θ-axis. On Γ the energy is $I \equiv -\cos a$ and $\dot{\theta}^2 = 2(\cos\theta - \cos a)$. By integration and the change of variables $\sin(\theta/2) = \sin(a/2)\sin\varphi$ we find

$$t = \int_0^{\varphi(t)} \frac{1}{\sqrt{1 - \sin^2(a/2)\sin^2 s}}\, ds$$

where $\theta(t)$ is the solution of the mathematical pendulum with the initial value

$$(\theta(0), \dot{\theta}(0)) = (0, 2\sin(a/2)).$$

Or, in terms of Jacobian elliptic functions, cf. [4, 20], where the elliptic modulus is $k := \sin(a/2)$, we find

$$\sin\varphi(t) = \mathrm{sn}[t, k]$$

and, using the trigonometric double angle formulas,

$$\cos\theta(t) = 1 - 2k^2\,\mathrm{sn}^2[t, k].$$

Also, the period of Γ is given by

$$4\int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 s}}\, ds = 4K(k)$$

where $K(k)$ is the complete elliptic integral of the first kind. Since, $t \mapsto \mathrm{sn}(t)$ has real period $4K$ (here and hereafter if the elliptic modulus is not given explicitly it is understood to be $k = \sin(a/2)$), the periodic orbit Γ is resonant when there are relatively prime positive integers K_1 and K_2 such that

$$K_1 \frac{2\pi}{\Omega} = K_2 4K(k).$$

Under this assumption and in view of the first order system

$$\begin{aligned}
\dot{z} &= -\Omega w, \\
\dot{w} &= \Omega z - \frac{\epsilon}{\Omega} g, \\
\dot{\theta} &= v, \\
\dot{v} &= -\sin\theta + \epsilon h,
\end{aligned}$$

the bifurcation function for a nonrotational resonance is given by

$$\mathcal{B}(p, q) = \left(\int_0^{K_1 2\pi/\Omega} wg\, dt, \quad \int_0^{K_1 2\pi/\Omega} vh\, dt \right)$$

where q is the coordinate on Γ introduced by using the solution $t \mapsto \theta(t + q)$, for $0 \le q < 4K(k)$. The components of the bifurcation function are computed as follows:

$$\begin{aligned}
\int_0^{K_1 2\pi/\Omega} wg\, dt &= K_1 \pi \lambda\, \mathrm{e}^{-2p} - A\, \mathrm{e}^{-p} I_1(q), \\
\int_0^{K_1 2\pi/\Omega} vh\, dt &= B\, \mathrm{e}^{-p} I_2(q) - I_3(q),
\end{aligned}$$

where

$$\begin{aligned}
I_1(q) &:= \int_0^{K_1 2\pi/\Omega} (\dot{\theta}(t+q))^2 \cos\theta(t+q) \sin\Omega t\, dt, \\
I_2(q) &:= \int_0^{K_1 2\pi/\Omega} \dot{\theta}(t+q) \sin\theta(t+q) \cos\Omega t\, dt, \\
I_3(q) &:= \int_0^{K_1 2\pi/\Omega} \dot{\theta}(t+q) M(\dot{\theta}(t+q))\, dt.
\end{aligned}$$

The integral I_3 depends on the static characteristic of the motor and the damping associated with the rotational motion as encoded in the function M. As a typical example and for definiteness in the computation we take M to be linear,

$$M(\dot{\theta}) := m_1 + m_2\dot{\theta};$$

more general model functions can be handled in a similar manner. For the linear case, we have the following proposition.

$$\begin{aligned}
I_3(q) &= \int_0^{K_1 2\pi/\Omega} m_1\dot{\theta} + m_2\dot{\theta}^2 \, dt \\
&= 2m_2 \int_0^{K_1 2\pi/\Omega} \cos\theta(t+q) - \cos a \, dt \\
&= -4m_2 K_1 \frac{\pi}{\Omega}\cos a + 2m_2 \int_0^{K_2 4K} \cos\theta(t)\, dt \\
&= -2m_2 K_2 4K \cos a + 2m_2 \int_0^{K_2 4K} 1 - 2k^2 \operatorname{sn}^2(t)\, dt.
\end{aligned}$$

The formula **310.02** of [4] can be used to evaluate the integral with integrand $\operatorname{sn}^2(t)$ to obtain

$$\begin{aligned}
I_3(q) &= -8m_2K_2(1+\cos a)K(k) + 4m_2 E(\operatorname{am}[K_2 4K(k), k], k) \\
&= -16m_2K_2(1-k^2)K(k) + 4m_2 E(\operatorname{am}[K_2 4K(k), k], k)
\end{aligned}$$

where $E(\varphi, k)$ denotes the normal elliptic integral of the second kind and $\operatorname{am}[u,k]$ is the amplitude, see [4]. Using [4, **113.02**, **122.06**], we obtain

$$E(\operatorname{am}[K_2 4K(k), k], k) = 4K_2 E(k)$$

where $E(k)$ is the complete elliptic integral of the second kind. Thus,

$$I_3(q) = 16m_2K_2(E(k) - (1-k^2)K(k)).$$

Note that for the linear static motor characteristic $q \mapsto I_3(q)$ is constant. Moreover,

$$I_3^* := \frac{1}{m_2} I_3(q) = \int_0^{K_1 2\pi/\Omega} \dot{\theta}^2 \, dt > 0.$$

For the integrals I_1 and I_2 we have the following identity.

IDENTITY 3.1.

$$\frac{3}{2}\Omega I_1(q) = (\cos a - \Omega^2) I_2(q).$$

Proof. Define $\eta := K_1 2\pi/\Omega$ and compute:

$$\begin{aligned}
I_1(q) &= \int_0^\eta 2(\cos\theta - \cos a)\cos\theta \sin\Omega t\, dt \\
&= 2\int_0^\eta \cos^2\theta \sin\Omega t\, dt - 2\cos a \int_0^\eta \cos a \sin\Omega t\, dt. \\
I_2(q) &= -\frac{1}{\Omega}\int_0^\eta (\ddot\theta \sin\theta + \dot\theta^2\cos\theta)\sin\Omega t\, dt \\
&= -\frac{1}{\Omega}\int_0^\eta \ddot\theta \sin\theta \sin\Omega t\, dt - \frac{1}{\Omega} I_1(q) \\
&= \frac{1}{\Omega}\int_0^\eta \sin^2\theta \sin\Omega t\, dt - \frac{1}{\Omega} I_1(q) \\
&= -\frac{1}{\Omega}\int_0^\eta \cos^2\theta \sin\Omega t\, dt - \frac{1}{\Omega} I_1(q) \\
&= -\frac{1}{\Omega}\left(\frac{1}{2} I_1(q) + \cos a \int_0^\eta \cos\theta \sin\Omega t\, dt\right) - \frac{1}{\Omega} I_1(q) \\
&= -\frac{3}{2\Omega} I_1(q) - \frac{\cos a}{\Omega}\left(-\int_0^\eta (-\dot\theta\sin\theta)(-\frac{1}{\Omega}\cos\Omega t)\, dt\right) \\
&= -\frac{3}{2\Omega} I_1(q) + \frac{\cos a}{\Omega^2} I_2(q). \square
\end{aligned}$$

Also, with the definition

$$I_c := \int_0^{K_1 2\pi/\Omega} \cos\theta(t)\cos\Omega t\, dt$$

we have a second identity.

Identity 3.2.

$$I_2(q) = \Omega I_c \sin\Omega q.$$

Proof. Define $\eta := K_1 2\pi/\Omega$ and compute:

$$\begin{aligned}
I_2(q) &= -\int_0^\eta \frac{d}{dt}(\cos\theta(t+q))\cos\Omega t\, dt \\
&= -\Omega\int_0^\eta \cos\theta(t+q)\sin\Omega t\, dt \\
&= -\Omega\int_0^\eta \cos\theta(t)\sin\Omega(t-q)\, dt \\
&= -\Omega\cos\Omega q\int_0^\eta \cos\theta(t)\sin\Omega t\, dt + \Omega\sin\Omega q\int_0^\eta \cos\theta(t)\cos\Omega t\, dt.
\end{aligned}$$

Since $t \mapsto \cos\theta(t)$ is an even function

$$\int_0^\eta \cos\theta(t)\sin\Omega t\, dt = 0. \square$$

Using the identities just obtained we have

$$\mathcal{B}(p,q) = \left(K_1\pi\lambda\,\mathrm{e}^{-2p} - \frac{2}{3}AI_c\,\mathrm{e}^{-p}(\cos a - \Omega^2)\sin\Omega q,\ -I_3 + BI_c\Omega\,\mathrm{e}^{-p}\sin\Omega q\right).$$

Thus, (p,q) is a zero of the bifurcation function if and only if this ordered pair is a solution of the bifurcation equations

$$K_1\pi\lambda - \frac{2}{3}AI_c(\cos a - \Omega^2)\,\mathrm{e}^p \sin\Omega q = 0,$$
$$I_3 - BI_c\Omega e^{-p}\sin\Omega q = 0.$$

Such a zero is simple provided

$$\frac{4}{3}ABI_c^2(\cos a - \Omega^2)\sin\Omega q\cos\Omega q \neq 0.$$

To show the bifurcation is nondegenerate we must show $I_c \neq 0$. It turns out that the validity of this condition depends on the resonance. This is the content of the following proposition.

PROPOSITION 3.3. *If K_1 and K_2 are relatively prime positive integers such that $K_1 2\pi/\Omega = K_2 4K(k)$, then for I_c to be nonvanishing it is necessary and sufficient that $K_2 = 1$ and $K_1 = 2n$ for some positive integer n. In case this condition holds*

$$I_c = 4\left(\frac{\pi^2 K_1}{K(k)}\right)\frac{\mathbf{q}^{K_1/2}}{1-\mathbf{q}^{K_1}} = 8\pi K_2\Omega\frac{\mathbf{q}^{K_1/2}}{(1-\mathbf{q}^{K_1})}$$

where $\mathbf{q} := e^{-\pi K'/K}$ is Jacobi's nome, [4, p. 11].

Proof. The proposition follows from the Fourier series representation of $u \mapsto \mathrm{sn}^2(u)$ given by

$$(kK)^2\,\mathrm{sn}^2(u) = K^2 - KE - 2\pi^2\sum_{n=1}^{\infty}\frac{n\mathbf{q}^n}{1-\mathbf{q}^{2n}}\cos 2nx$$

where $x := \pi u/(2K)$. (This formula is stated without proof in [20, p. 520]. A second Fourier series expansion in [4, **911.01**] seems to be incorrect. Thus, even though a reference for the formula exists, we will verify this series representation below.) Define $\eta := K_1 2\pi/\Omega$. To prove the proposition, compute

$$\begin{aligned}
I_c &= 4\left(\frac{\pi}{K}\right)^2\sum_{n=1}^{\infty}\frac{n\mathbf{q}^n}{1-\mathbf{q}^{2n}}\int_0^{\eta}\cos\left(\frac{n\pi}{K}u\right)\cos\Omega u\,du\\
&= 4\left(\frac{\pi}{K}\right)^2\sum_{n=1}^{\infty}\frac{n\mathbf{q}^n}{1-\mathbf{q}^{2n}}\int_0^{\eta}\cos\left(2n\frac{K_2}{K_1}\Omega u\right)\cos\Omega u\,du.
\end{aligned}$$

After the substitution $v := \Omega u/K_1$, we obtain

$$I_c = 4\left(\frac{\pi}{K}\right)^2\sum_{n=1}^{\infty}\frac{n\mathbf{q}^n}{1-\mathbf{q}^{2n}}\frac{K_1}{\Omega}\int_0^{2\pi}\cos 2nK_2 v\cos K_1 v\,dv.$$

Thus, I_c vanishes unless $2nK_2 = K_1$. In particular, K_1 must be even and K_2 must be a factor of K_1. Since K_1 and K_2 are relatively prime, $K_2 = 1$. If $I_c \neq 0$, then

$$\begin{aligned}
I_c &= 4\left(\frac{\pi}{K}\right)^2\left(\frac{K_1}{\Omega}\right)\left(\frac{K_1}{2}\right)\frac{\mathbf{q}^{K_1/2}}{1-\mathbf{q}^{K_1}}\pi\\
&= 4\frac{\pi^2 K_1}{K}\frac{\mathbf{q}^{K_1/2}}{1-\mathbf{q}^{K_1}}
\end{aligned}$$

as required.

To verify the Fourier series expansion we compute the value of

$$J := \int_{-\pi}^{\pi} \mathrm{sn}^2\left(\frac{2K}{\pi}x\right) \mathrm{e}^{imx}\, dx, \qquad m \neq 0$$

by contour integration around the parallelogram in the complex plane with vertices $-\pi$, π, $\pi\tau$ and $\pi\tau - 2\pi$ where $\tau := iK'/K$, cf. [20, p. 510]. Using the fact that $u \mapsto \mathrm{sn}(u)$ is doubly periodic with periods $4K$ and $2iK'$ and $x \mapsto \mathrm{e}^{imx}$ is periodic with period 2π, the path integrals along the edges of the parallelogram given by $[\pi, \pi\tau]$ and $[\pi\tau - 2\pi, -\pi]$ cancel. Also, an easy computation shows the integral along the edge $[\pi\tau, \pi\tau - 2\pi]$ is $-\mathrm{e}^{im\pi\tau}\,\mathrm{e}^{im\pi} J$. Thus,

$$\left(1 - \mathrm{e}^{im\pi\tau}\,\mathrm{e}^{im\pi}\right) J = 2\pi i \sum(\text{residues}).$$

The poles of $u \mapsto \mathrm{sn}(u)$ reside at the points in the complex plane congruent to iK' and $2K + iK'$ modulo the periods of sn. It follows that $\mathrm{sn}^2(2Kx/\pi)\,\mathrm{e}^{imx}$ has exactly two poles in the parallelogram. These poles are at the points $\pi\tau/2$ and $\pi\tau/2 - \pi$. To compute the residues, start with the Maclaurin series for $u \mapsto \mathrm{sn}(u)$ given by

$$\mathrm{sn}(u) = u + O(u^3)$$

and the identity

$$\mathrm{sn}(u + iK') = \frac{1}{k\,\mathrm{sn}(u)}$$

to obtain

$$\mathrm{sn}(u + iK') = \frac{1}{ku} + O(u).$$

Set $u + iK' = 2Kx/\pi$ to get

$$\mathrm{sn}(\frac{2K}{\pi}x) = \frac{\pi}{2kK(x - \pi\tau/2)} + O(x - \pi\tau/2)$$

and compute

$$\mathrm{sn}^2(\frac{2K}{\pi}x)\,\mathrm{e}^{imx} = \left(\frac{\pi}{2kK}\right)^2 \frac{\mathrm{e}^{im\pi\tau/2}}{(x - \pi\tau/2)^2} + \left(\frac{\pi}{2kK}\right)^2 \frac{im\,\mathrm{e}^{im\pi\tau/2}}{(x - \pi\tau/2)} + O(1).$$

Thus, the residue at $\pi\tau/2$ is

$$\left(\frac{\pi}{2kK}\right)^2 im\,\mathrm{e}^{im\pi\tau/2} = \left(\frac{\pi}{2kK}\right)^2 im\mathbf{q}^{m/2}.$$

Use the identity

$$\mathrm{sn}(u - 2K + iK') = -\mathrm{sn}(u + iK')$$

and a similar computation to compute the residue at $-\pi + \pi\tau/2$. We find this residue to be

$$\left(\frac{\pi}{2kK}\right)^2 im\,\mathrm{e}^{-im\pi}\mathbf{q}^{m/2}.$$

From this it follows that

$$J = -\frac{\pi^3}{2(kK)^2}\frac{m\mathbf{q}^{m/2}}{1-\mathbf{q}^m\,\mathrm{e}^{im\pi}}\left(1+\mathrm{e}^{-im\pi}\right).$$

Thus, the Fourier coefficient corresponding to J vanishes unless $m = 2n$ in which case

$$J = -2\frac{\pi^3}{(kK)^2}\frac{n\mathbf{q}^n}{1-\mathbf{q}^{2n}}.$$

Since $x \mapsto \mathrm{sn}^2(2Kx/\pi)$ is even, its Fourier series is a cosine series. In fact, the Fourier coefficient of $\cos 2nx$ is the real part of J/π. Since J is real,

$$\mathrm{sn}^2(\frac{2K}{\pi}x) = a_0 - 2\left(\frac{\pi}{kK}\right)^2\sum_{n=1}^{\infty}\frac{n\mathbf{q}^n}{1-\mathbf{q}^{2n}}\cos 2nx.$$

The constant term a_0 can be shown to agree with the stated formula. But, since we do not require its value here, the proof is left to the reader. □

By the proposition we see there are (under appropriate choices of the constant parameters) bifurcating families of periodic solutions for the full model equations at each nonrotational periodic motion, of the gravitationally influenced rotor, whose period is an even multiple of the natural period of the support oscillator. In fact, if we impose the nondegeneracy conditions $K_1 = 2n$ and $K_2 = 1$, then, by eliminating $\sin\Omega q$ from the bifurcation equations, we find

$$\mathrm{e}^{2p} = \left(\frac{3\pi\lambda B\Omega}{A}\right)\frac{n}{m_2 I_3^*(\cos a - \Omega^2)}.$$

Thus, we can solve for p provided $m_2(\cos a - \Omega^2) > 0$. Assuming this condition is satisfied and inserting e^p into the second bifurcation equation we find there are two solutions for q provided $-1 < \Delta < 1$ for

$$\Delta := \left(\frac{3n\pi\lambda}{AB\Omega}\right)^{1/2}\left(\frac{I_3^*}{m_2(\cos a - \Omega^2)}\right)^{1/2}\left(\frac{m_2}{I_c}\right).$$

The question arises as to how many resonant periodic solutions of the unperturbed mathematical pendulum correspond to nondegenerate bifurcation points. It is clearly possible to obtain any preassigned finite number of simultaneous bifurcations. However, it is not possible to have infinitely many. To have infinitely many bifurcation points for a fixed set of parameter values it is necessary that Δ remain bounded in the unit interval for infinitely many integers n such that $n = \Omega K(k)/\pi$. To show this is not the case, note $k \to 1$ as $n \to \infty$, and use the computations made above for I_3^* and I_c together with the fact that $\cos a = 1 - 2k^2$ to compute

$$\Delta = -\sqrt{n}\,m_2 k^2\left(\frac{3\lambda}{16AB\Omega^3\pi}\right)^{1/2}\left(\frac{1-\mathbf{q}^{2n}}{\mathbf{q}^n}\right)\left(\frac{I_3^*(k)}{m_2(1-2k^2-\Omega^2)}\right)^{1/2}.$$

We claim Δ grows without bound as $k \to 1$. Note first that as $k \to 1$,

$$\mathbf{q}^n = \mathrm{e}^{-\Omega K(\sqrt{1-k^2})} \to \mathrm{e}^{-\Omega\pi/2}$$

so the term $(1-\mathbf{q}^{2n})/(\mathbf{q}^n)$ remains bounded. Also, as $k \to 1$ we have $K(k) - \ln(4/\sqrt{1-k^2}) \to 0$. Using these facts and the expression for $I_3^*(k)$ it follows that $I_3^*(k)$

remains bounded as $k \to 1$. Thus, all terms except the $\sqrt{n}$ term remain bounded. It follows that $\Delta \to \infty$ as $k \to 1$ and $n \to \infty$. However, the fact that infinitely many different resonances can lead to nondegenerate first order bifurcation to periodic solutions is in marked contrast to the case when the gravitational forces are considered small and the only nondegenerate bifurcation occurs for the primary resonance.

To analyze the rotational motion of the rotor, recall the mathematical pendulum system defined on the phase plane is given by

$$\dot{\theta} = v, \qquad \dot{v} = -\sin\theta + \epsilon h.$$

The rotational motions are naturally defined on the phase cylinder that is obtained from the phase plane by viewing the variable θ modulo 2π. There are two families of periodic solutions corresponding to $\dot{\theta} < 0$ and $\dot{\theta} > 0$. For definiteness we will treat the case $\dot{\theta} < 0$, the other case is similar. In particular, since we have changed the coordinates of the model equation by $\theta \to -\theta - \pi$, a positive rotation in the original model equations corresponds to a negative rotation here. It is convenient to choose the (symplectic) coordinate chart on the phase cylinder given by the transformations $x = \sqrt{-v}\cos\theta$, $y = \sqrt{-v}\sin\theta$. The chart for the second case would be $x = \sqrt{v}\sin\theta$, $y = \sqrt{v}\cos\theta$. This choice of coordinates ensures the divergence of the transformed vector field vanishes and that the function $\beta_2(t,p)$ defined in §3 is zero. In the (x,y)-plane, the phase plane system becomes

$$\begin{aligned}
\dot{x} &= y(x^2+y^2) + \frac{1}{2}xy(x^2+y^2)^{-3/2} - \frac{\epsilon}{2}\frac{x}{x^2+y^2}h, \\
\dot{y} &= -x(x^2+y^2) + \frac{1}{2}y^2(x^2+y^2)^{-3/2} - \frac{\epsilon}{2}\frac{y}{x^2+y^2}h,
\end{aligned}$$

where

$$h = -\Big(M(\dot{\theta}) - Bz\sin\theta\Big) = -\left(M(-(x^2+y^2)) - B\frac{yz}{\sqrt{x^2+y^2}}\right).$$

We study the above system coupled as before to the support oscillator given by

$$\dot{z} = -\Omega w, \qquad \dot{w} = \Omega z - \frac{\epsilon}{\Omega}g$$

where

$$\begin{aligned}
g &= -\Big(F(z) + \lambda\dot{z} + A\dot{\theta}^2\cos\theta\Big) \\
&= -\Big(F(z) - \lambda\Omega w + Ax(x^2+y^2)^{3/2}\Big).
\end{aligned}$$

Since the rotational motions correspond to curves in the phase plane which do not intersect the θ axis, it is convenient to consider the v axis as a section for the flow. On the trajectory passing through the point in the phase plane with coordinates $(0,b)$, $|b| > 2$ the first integral $I := v^2/2 - \cos\theta$ has the constant value $I \equiv b^2/2 - 1$. The case $\dot{\theta} < 0$ corresponds to $b < 2$ and we have

$$\frac{1}{2}\dot{\theta}^2 = \cos\theta + \frac{1}{2}b^2 - 1.$$

Define $\varphi := \theta/2$ and $k := 2/|b|$ to obtain equivalently

$$\dot{\varphi}^2 = \frac{1}{k^2}\left(1 - k^2 \sin^2 \varphi\right)$$

so that

$$\int_0^{\varphi(t)} \frac{1}{\sqrt{1 - k^2 \sin^2 s}}\, ds = \operatorname{sgn}(\mathrm{b}) \frac{t}{k}.$$

In terms of Jacobian elliptic functions, we have

$$\begin{aligned} \varphi(t) &= \operatorname{sgn}(\mathrm{b}) \operatorname{am}[t/k, k], & \dot{\varphi}(t) &= \operatorname{sgn}(\mathrm{b}) \operatorname{dn}[t/k, k]/k, \\ \cos \varphi(t) &= \operatorname{cn}[t/k, k], & \sin \varphi(t) &= \operatorname{sgn}(\mathrm{b}) \operatorname{sn}[t/k, k] \end{aligned}$$

or, using the trigonometric double angle formulas,

$$\begin{aligned} \theta(t) &= 2\operatorname{sgn}(\mathrm{b}) \operatorname{am}[t/k, k], & \dot{\theta}(t) &= 2\operatorname{sgn}(\mathrm{b}) \operatorname{dn}[t/k, k]/k, \\ \cos \theta(t) &= 1 - 2 \operatorname{sn}^2[t/k, k], & \sin \theta(t) &= 2\operatorname{sgn}(\mathrm{b}) \operatorname{sn}[t/k, k] \operatorname{cn}[t/k, k]. \end{aligned}$$

Using these formulas, the definition of the phase cylinder, and $b < 0$, we have

$$\begin{aligned} x(t) &= \left(\frac{2}{k} \operatorname{dn}[t/k, k]\right)^{1/2} \left(1 - 2 \operatorname{sn}^2[t/k, k]\right), \\ y(t) &= \left(\frac{2}{k} \operatorname{dn}[t/k, k]\right)^{1/2} 2 \operatorname{sn}[t/k, k] \operatorname{cn}[t/k, k]. \end{aligned}$$

Also, observe the period T of the periodic solution on the phase cylinder with initial value $(0, b)$ is given by $\theta(T/2) = \operatorname{sgn}(\mathrm{b})\pi$. Thus,

$$T = 2k \operatorname{am}^{-1}[\pi/2, k] = 2kK(k)$$

and the resonance relation is given by

$$K_1 \frac{2\pi}{\Omega} = K_2 2kK(k).$$

Using the results of §3,

$$\mathcal{B} = \left(\int_0^{K_1 2\pi/\Omega} wg\, dt, \quad -\frac{1}{2} \int_0^{K_1 2\pi/\Omega} (x^2 + y^2) h\, dt \right).$$

It is preferable initially to express the components of $\mathcal{B}$ in phase plane coordinates:

$$\begin{aligned} \int_0^{K_1 2\pi/\Omega} wg\, dt &= K_1 \pi \lambda \,\mathrm{e}^{-2p} - A \,\mathrm{e}^{-p} I_1^r(q), \\ -\frac{1}{2} \int_0^{K_1 2\pi/\Omega} (x^2 + y^2) h\, dt &= -\frac{1}{2} I_3^r(q) + \frac{1}{2} B \,\mathrm{e}^{-p} I_2^r(q), \end{aligned}$$

where

$$\begin{aligned} I_1^r(q) &:= \int_0^{K_1 2\pi/\Omega} (\dot{\theta}(t+q))^2 \cos \theta(t+q) \sin \Omega t\, dt, \\ I_2^r(q) &:= \int_0^{K_1 2\pi/\Omega} \dot{\theta}(t+q) \sin \theta(t+q) \cos \Omega t\, dt, \\ I_3^r(q) &:= \int_0^{K_1 2\pi/\Omega} \dot{\theta}(t+q) M(\dot{\theta}(t+q))\, dt. \end{aligned}$$

Also, we define

$$I_c^r(q) := \int_0^{K_1 2\pi/\Omega} \cos\theta(t)\cos\Omega t\, dt.$$

As in the case of the nonrotational motions, we have the following identities:

IDENTITY 3.4.

$$\begin{aligned}\frac{3}{2}\Omega I_1^r(q) &= -\left(\frac{2}{k^2} - 1 + \Omega^2\right) I_2^r(q),\\ I_2^r(q) &= \Omega I_c^r \sin\Omega q.\end{aligned}$$

Using these identities, we find

$$\mathcal{B}(p,q) := (\mathcal{B}_1(p,q), \mathcal{B}_2(p,q))$$

where

$$\begin{aligned}\mathcal{B}_1(p,q) &= K_1\pi\lambda\,\mathrm{e}^{-2p} + \frac{2}{3k^2}A\,\mathrm{e}^{-p}\left(2 + k^2(\Omega^2 - 1)\right) I_c^r \sin\Omega q,\\ \mathcal{B}_2(p,q) &= -\frac{1}{2}I_3^r + \frac{1}{2}B\,\mathrm{e}^{-p}\Omega I_c^r \sin\Omega q.\end{aligned}$$

PROPOSITION 3.5. *If K_1 and K_2 are relatively prime positive integers such that $K_1 2\pi/\Omega = K_2 2kK(k)$, then for I_c^r to be nonvanishing it is necessary and sufficient that $K_2 = 1$. In case this condition holds*

$$I_c^r = 4\left(\frac{\pi^2 K_1}{kK(k)}\right)\frac{\mathbf{q}^{K_1}}{1 - \mathbf{q}^{2K_1}} = 4\pi\Omega\frac{\mathbf{q}^{K_1}}{1 - \mathbf{q}^{2K_1}}$$

where $\mathbf{q} := e^{-\pi K'/K}$ is Jacobi's nome.

Proof. The integral I_c^r is computed as in proposition 3.3 using the Fourier series for $\mathrm{sn}^2(u)$. In fact,

$$\begin{aligned}I_c^r &= -2\int_0^{K_1 2\pi/\Omega} \mathrm{sn}^2(t/k)\cos\Omega t\, dt\\ &= 4\left(\frac{\pi}{kK}\right)^2 \sum_{n=1}^{\infty}\frac{n\mathbf{q}^n}{1 - \mathbf{q}^{2n}}\int_0^{K_1 2\pi/\Omega}\cos\left(\frac{n\pi}{kK}t\right)\cos\Omega t\, dt.\end{aligned}$$

After the change of variables $v := \Omega t/K_1$ and substitution from the resonance relation we obtain

$$I_c^r = 4\left(\frac{\pi}{kK}\right)^2\frac{kKK_2}{\pi}\sum_{n=1}^{\infty}\frac{n\mathbf{q}^n}{1 - \mathbf{q}^{2n}}\int_0^{2\pi}\cos nK_2 v\cos K_1 v\, dv.$$

Thus, I_c^r vanishes unless $nK_2 = K_1$. Since K_1 and K_2 are relatively prime, this means $I_c^r \neq 0$ exactly when $K_2 = 1$ and K_1 is arbitrary. In this case we obtain

$$I_c^r = 4\left(\frac{\pi^2 K_1}{kK(k)}\right)\frac{\mathbf{q}^{K_1}}{1 - \mathbf{q}^{2K_1}}.$$

□

Finally, we compute I_3^r under the assumption $M(\dot{\theta}) = m_1 + m_2\dot{\theta}$. For this we have

$$I_3^r = m_1 \int_0^{K_1 2\pi/\Omega} \dot{\theta}(t+q)\,dt + m_2 \int_0^{K_1 2\pi/\Omega} \dot{\theta}^2(t+q)\,dt.$$

In the present case we find, using the resonance relation and the periodicity,

$$\int_0^{K_1 2\pi/\Omega} \dot{\theta}(t+q)\,dt = -2\pi K_2.$$

Also, as before,

$$\begin{aligned}\int_0^{K_1 2\pi/\Omega} \dot{\theta}^2(t+q)\,dt &= \int_0^{K_1 2\pi/\Omega} (2\cos(\theta(t+q)) + b^2 - 2)\,dt \\ &= (b^2-2)K_1\frac{2\pi}{\Omega} + 2\int_0^{K_1 2\pi/\Omega} 1 - 2\mathrm{sn}^2[t/k,k]\,dt \\ &= \frac{4}{k}E(\mathrm{am}\,[2K_2K(k),k],k) \\ &= 8K_2\frac{E(k)}{k}\end{aligned}$$

Thus, we have

$$-\frac{1}{2}I_3^r(q) = m_1\pi K_2 - m_2 4K_2\frac{E(k)}{k}.$$

By the proposition we see there are (under appropriate choices of the constant parameters) bifurcating families of periodic solutions for the full model equations at each rotational periodic motion, of the gravitationally influenced rotor, whose period is an integer multiple of the natural period of the support oscillator. The fact that the resonances are not restricted to *even* multiples of the period of the support oscillator as in the case of nonrotational motions is perhaps expected since near the separatrix between rotational and nonrotational motions the nonrotational periods are twice as long as the rotational periods. More precisely, if we impose the nondegeneracy condition $K_2 = 1$, the bifurcation points are the simple solutions of the equations

$$\begin{aligned}\lambda K_1\pi e^{-2p} + \frac{2A}{3k^2}(2+k^2(\Omega^2-1))4\pi\Omega\frac{\mathbf{q}^{K_1}}{1-\mathbf{q}^{2K_1}}e^{-p}\sin\Omega q = 0,\\ m_1\pi - m_2 4\frac{E(k)}{k} + 2\pi B\Omega^2\frac{\mathbf{q}^{K_1}}{1-\mathbf{q}^{2K_1}}e^{-p}\sin\Omega q = 0.\end{aligned}$$

By eliminating $\sin\Omega q$ from these equations, we find

$$\mathrm{e}^{2p} = \Delta := \frac{3\pi\lambda B\Omega K_1 k^3}{4A(m_1\pi k - m_2 4E(k))(2+k^2(\Omega^2-1))}.$$

Thus, we can solve for p provided $(m_1\pi k - m_2 4E(k))(2+k^2(\Omega^2-1)) > 0$. Assuming this condition is satisfied and inserting e^p into the second bifurcation equation we find $\sin\Omega q := \Lambda$ where

$$\Lambda = -\left(\frac{3\pi\lambda B\Omega K_1 k^3}{4A(m_1\pi k - m_2 4E(k))(2+k^2(\Omega^2-1))}\right)^{1/2}\left(\frac{m_1\pi k - m_2 4E(k)}{2\pi B\Omega^2 k}\right)\frac{1-\mathbf{q}^{2K_1}}{\mathbf{q}^{K_1}}.$$

Thus, there are two solutions for q provided $-1 < \Lambda < 1$. In addition, it is easy to compute the Jacobian of the two bifurcation equations and deduce that the solutions of the bifurcation equations will both be simple provided $\cos \Omega q \neq 0$. This is as it should be since the solutions are simple when there are two values of q and not simple at the bifurcation points given by $\sin \Omega q = \pm 1$.

As in the case of the nonrotational motions, if the parameters are fixed, then there are only finitely many resonant motions of the rotor for which the condition $-1 < \Lambda < 1$ is satisfied. This follows as before by showing Λ is unbounded as $k \to 1$ and $K_1 \to \infty$. Thus, again for rotational motions under a strong gravitational force, infinitely many resonant solutions can lead to nondegenerate first order bifurcation, but only a finite number of these are excited for a fixed set of parameter values.

We end this section with a useful observation. The divergence of the perturbed vector field, computed in (z, w, x, y)-coordinates, is constant. In fact, the divergence is simply $-\epsilon(\lambda + m_2)$. This is reasonable since λ and m_2 are coefficients of damping in the system. Abel's formula applied to the linear variational equations as in [19, p. 156] implies the determinant of the linearized Poincaré map is given by

$$\det DP(\xi, \epsilon) = e^{-\epsilon(\lambda + m_2) K_1 2\pi/\Omega}.$$

Thus, the linearized Poincaré map contracts volume and the perturbed periodic solutions found by our bifurcation method are all saddles and sinks. In particular, this shows entrainment (capture) is possible.

3.1. Remarks, Experiments and Speculation. We have just shown there exist choices of the parameters in our model equations such that several periodic solutions, corresponding to rotational motions of the rotor, can coexistent. Moreover, these periodic solutions in the four dimensional phase space are all saddles or sinks. In order to determine the dynamics of the system, we would like further stability information about these periodic solutions. Rigorous stability information may be obtained from a second order bifurcation analysis. However, we mention that the bifurcating families occur in pairs, corresponding to the solutions of the equation $\sin \Omega q = \Lambda$. Generically, one bifurcating family consists of sinks the other consists of saddles. The basin of attraction of a periodic solution corresponding to a sink is the region in phase space "captured into resonance" or, in other language, it is the entrainment domain. Of course, there is no obvious reason why such a periodic solution will be globally attracting, thus solutions starting outside of the basin of attraction have a different fate. On the other hand, a saddle periodic solution may have a one, two or three dimensional stable manifold. Solutions starting near the stable manifold may remain near the saddle periodic solution on a very long time scale, appearing to be captured, only to eventually leave the vicinity of the saddle periodic solution along its unstable manifold to pass near a second saddle or perhaps become entrained to a stable periodic solution. If there are several such saddles, this behavior may be very complex.

At the end of the last section we showed the linearized Poincaré map contracts volume. This fact was used to prove the perturbed periodic solutions are saddles and sinks. In contrast to the similar analysis of a single forced oscillator, e.g. [19, p. 157] or [9, p. 207], we can not conclude there are no invariant closed curves for the *three* dimensional perturbed Poincaré map. In other words, periodic sinks may coexists with more complicated attractors. Before discussing this possibility more fully we mention that the analysis completed above only considers the bifurcation of periodic solutions from periodic solutions of the unperturbed oscillators at resonance.

In the example with a strong gravitational force, the mathematical pendulum has, in the phase cylinder, a hyperbolic saddle point corresponding to its unstable equilibrium state, and this rest point has a pair of associated homoclinic trajectories. The dynamics of the perturbed system near the corresponding trajectories in the four dimensional phase space of the coupled system can perhaps be determined to some extent by analyzing an appropriate "Melnikov" integral. Such an analysis might show the presence of horseshoes. In any case, the existence of complicated attractors remains to be established.

As an excursion in this direction, we have considered a decoupled specialization of our model equations in order to obtain a two dimensional Poincaré map and the possibility of visual representations of some aspects of the dynamics. For this, we consider the system

$$\begin{aligned}&\ddot{z}+\Omega^2 z=0,\\&\ddot{\theta}+\sin\theta=-\epsilon(m_1+m_2\dot{\theta}-Bz\sin\theta).\end{aligned}$$

It may be viewed as a single parametrically excited mathematical pendulum.

As before, to study the rotational motions, we consider (symplectic) polar coordinates on the phase cylinder to obtain

$$\begin{aligned}\dot{x}&=y(x^2+y^2)+\frac{1}{2}xy(x^2+y^2)^{-3/2}-\frac{\epsilon}{2}\frac{x}{x^2+y^2}h,\\\dot{y}&=-x(x^2+y^2)+\frac{1}{2}y^2(x^2+y^2)^{-3/2}-\frac{\epsilon}{2}\frac{y}{x^2+y^2}h,\end{aligned}$$

where

$$h=-m_1+m_2(x^2+y^2)+B\frac{y}{\sqrt{x^2+y^2}}\mathrm{e}^{-p}\cos\Omega t.$$

A comparison of the analysis for the coupled system with the analysis for the single "forced" oscillator as presented in [5, 6] shows we have already computed the bifurcation function for this system. Here, p is just a parameter, and the scalar bifurcation function is just $\mathcal{B}_2(q)$ as computed above. In fact, for the $(K_1 : K_2)$ resonance, the bifurcation equation is

$$\mathcal{B}_2(q)=\begin{cases}m_1\pi-m_2 4\frac{E(k)}{k}+2\pi B\Omega^2\frac{\mathbf{q}^{K_1}}{1-\mathbf{q}^{2K_1}}e^{-p}\sin\Omega q=0, & \text{if } K_2=1\\ K_2(m_1\pi-m_2 4\frac{E(k)}{k}), & \text{if } K_2\neq 1.\end{cases}$$

In case $K_2=1$, the bifurcation equation $\mathcal{B}_2(q)=0$ is equivalent to $\sin\Omega q:=\Lambda$ where

$$\Lambda=-\mathrm{e}^{p}\left(\frac{m_1\pi k-m_2 4E(k)}{2\pi B\Omega^2 k}\right)\frac{1-\mathbf{q}^{2K_1}}{\mathbf{q}^{K_1}}.$$

Thus, there are two solutions for q provided $-1<\Lambda<1$. Here, the linearized Poincaré map is still area contracting (the divergence is $-\epsilon m_2$) so the periodic solutions are again saddles and sinks. However, even in this case we *can not* conclude there are no invariant curves in the Poincaré section. This is due to the fact that, for the rotational motions, the system is defined on an annulus in the phase cylinder whose inner boundary is the separatrix of the unperturbed mathematical pendulum. This

FIG. 3.1. *Schematic representation of basin boundaries for attractors in the perturbed Poincaré map for the rotational motions of the parametrically excited mathematical pendulum. Shaded region is in the basin of attraction of the invariant torus.*

fact is reflected in the singularity of the (x, y)-coordinates at $x^2 + y^2 = -\dot{\theta} = 0$. In other words, in the (x, y) section the region corresponding to rotational motion is an annular region surrounding the origin. More precisely, the unperturbed rotational solutions correspond to solutions (outside the separatrices) with energies

$$\begin{aligned} E &= \frac{1}{2}\dot{\theta}^2 + 1 - \cos\theta \\ &= \frac{1}{2}(x^2 + y^2)^2 + 1 - \frac{x}{\sqrt{x^2 + y^2}} > 1. \end{aligned}$$

When the system is perturbed, solutions can cross into the region with $E < 1$ and then eventually cross the curve $\dot{\theta} = 0$ where the vector field is singular. Thus, the area of the region corresponding to rotational motions is not preserved. Of course, the fact that the linearized Poincaré map is area contracting does imply there is at most one invariant curve. If there were two invariant curves, the annular region bounded by these curves would be invariant. Numerical experiments suggest that in fact invariant curves exit. This suggests a similar phenomenon is possible for the coupled system. But, at present we do not know how to examine this possibility rigorously.

We have investigated the dynamics of the uncoupled system in the region of parameter space corresponding to parameter values where the *coupled* system has periodic solutions arising from the bifurcation theory given previously. A useful example is provided by the following choice of parameters:

$$K_1 = 1, \quad \Omega = 4, \quad A = 4, \quad \lambda = 0, \quad B = 4, \quad m_1 = 10$$

with m_2 and p variable.

Let Γ denote the $(K_1 : K_2) = (1 : 1)$ resonant periodic solution of the mathematical pendulum. This solution is given by the elliptic modulus $k_* \approx 0.47$. Or, in the original coordinates, it is the solution starting at $(\theta, \dot{\theta}) = (0, -2/k_*)$. Recall a necessary condition for the bifurcation equations obtained for the *coupled* system to have solutions in this case is $m_1 \pi k_* - m_2 4E(k_*) > 0$. Thus, for the given parameters, we must have $0 < m_2 < 5\pi k_*/(2E(k_*)) \approx 2.495$. If this condition is satisfied, p is determined by the formula $e^{2p} = \Delta$ given previously.

The unperturbed Poincaré map for the uncoupled system giving the return to the (x, y)-plane after time $2\pi/\Omega$ has Γ as an invariant curve. In fact, the unperturbed Poincaré map is the identity on Γ. In addition, for small positive ϵ, we have proved there are two fixed points for the perturbed Poincaré map near the zeros of $\mathcal{B}_2(q)$ and that these fixed points correspond to the persistent periodic solutions. For the resonances with $K_2 \neq 1$, the bifurcation function reduces to $b(k) := K_2(m_1\pi - m_2 4E(k)/k)$ and is independent of q. It is easy to see the dense set of resonant orbits such that $K_1/K_2 < 1$ lie "outside" Γ in the Poincaré section while the dense set of resonant orbits such that $K_2 \neq 1$ and $K_1/K_2 > 1$ lies "inside" Γ. Moreover, the reduced bifurcation function b is positive inside and on Γ. That is, there is an unperturbed periodic solution Γ_0 of the mathematical pendulum corresponding to some $k_0(m_2) < k_*$ such that $b(k_0) = 0$. In particular, Γ_0 surrounds Γ, $b(k) < 0$ for all $k < k_0$ and the resonant orbits outside Γ_0 correspond to resonances such that $K_1/K_2 < 1$. Thus, for these resonant orbits $K_2 \neq 1$ and $\mathcal{B}_2 = b$.

In general, some of the resonant orbits corresponding to $K_2 = 1$ and $K_1 > 1$ can be excited and additional periodic solutions can occur. But, for our choice of parameters this does not happen. Thus, in a similar manner to the discussion in [19, pp 161-175], we observe that perturbed trajectories of the Poincaré map tend to drift outward toward Γ_0 from the region inside Γ_0, except for the resonance layer near Γ, and they tend to drift inward toward Γ_0 from the outside. In particular, there are no periodic solutions excited by the perturbation except for those on Γ.

The existence of a periodic sink and a periodic saddle for the perturbed Poincaré map corresponding to the perturbed periodic solutions is consistent with these facts. This is exactly the situation observed in numerical simulation. In addition, the positions of the bifurcation points as predicted independently by solving the bifurcation equations $e^{2p} = \Delta$ and $\sin \Omega q = \Lambda$ is also confirmed. However, the facts about the sign of b and the implied drift directions for the perturbed solutions indicate there is also the possibility of a non periodic attractor Γ_ϵ near Γ_0 coexistent with the periodic sink. Our numerical experiments confirm the existence of such an invariant attracting set. It appears to be a smooth curve for $0 < m_2 < 2.459$ and for all sufficiently small $\epsilon > 0$.

From the discussion above, the periodic sink lies inside the region bounded by this invariant curve. Because there are two attractors, the entrainment domain (the basin of attraction of the periodic sink) shares a common boundary with the basin of attraction of the invariant curve. Fig. 1 shows, schematically, the basins of attraction for the two attractors. Aside from the fact that the spiral basin of attraction of the periodic sink is very thin for small ϵ, we also see that solutions with initial values "outside" Γ_ϵ are never entrained to the periodic sink. We expect the entrainment domain for the coupled system to be at least as complex.

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