Inertial flows, slow flows, and combinatorial identities for delay equations

Carmen Chicone[†]

August 14, 2004

Abstract

The vector field induced on the finite-dimensional inertial manifold of a delay equation with small delay is proved to agree, up to the order of the expansion, with the vector field induced on a slow manifold of the differential equation obtained from the delay equation by expanding to some finite order in powers of the delay. In addition, the smoothness of inertial vector fields, the smoothness of slow vector fields, and the existence of combinatorial-style identities obtained by equating the series expansions of the slow and inertial vector fields are discussed.

Keywords: Delay equation, Inertial manifold, Slow manifold
AMS Classification: 34K19

1 Introduction

The purpose of this paper is to prove a conjecture stated in [2] concerning delay equations of the form

$$\dot{x}(t) = f(x(t), x(t-\tau)), \tag{1}$$

[†] Department of Mathematics, University of Missouri, Columbia, MO 65211.
E-mail: carmen@chicone.math.missouri.edu

Research supported in part by a grant from the University of Missouri Research Board.

where $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is C^∞ and globally Lipschitz. For each sufficiently small $\tau > 0$, this delay equation has an n-dimensional smooth inertial manifold in its infinite-dimensional phase space $C([-\tau, 0], \mathbb{R}^n)$. Also, the Nth-order ordinary differential equation obtained by replacing the right-hand side of the delay equation by the Nth-order Taylor polynomial centered at $\tau = 0$ of the function $\tau \mapsto f(x(t), x(t-\tau))$, when viewed as an appropriate singularly perturbed first-order system, has an n-dimensional slow manifold in its nN-dimensional state space. We will show that the vector fields obtained by restriction to these inertial and slow manifolds agree to order $N-1$ in τ under the additional assumption that the derivative $D_2 f(x, x)$ is invertible for all x, thus proving a version of the conjecture stated in [2]. The motivation for this conjecture presumes some familiarity with the discussion in [2]. We will outline the main ideas to help orient the reader.

Delay equations like (1) often arise in physics associated with forces that are not transmitted instantaneously, but with a delay (such as the time required for light to travel from one point to another). The space of all solutions of a delay equation is infinite-dimensional, which is often not physically appropriate. On the other hand, typically there is a physically appropriate n-dimensional submanifold of solutions, called an *inertial manifold*, with two important properties: the inertial manifold is invariant and it is a global attractor. The latter means that every solution in the infinite-dimensional state space of the delay equation approaches the inertial manifold (exponentially fast in this case) as time increases to infinity. Thus, the dynamical system restricted to an inertial manifold models the long-term behavior of the physical system that the delay equation (1) describes.

The inertial manifold for equation (1) with a sufficiently small delay is parametrized by $\mathbb{R}^n$ in the following way. For each vector $\xi \in \mathbb{R}^n$, the inertial manifold contains a unique solution $t \mapsto x(t)$ of the delay equation that satisfies the initial condition $x(\theta) = \xi$ for $-\tau \le \theta \le 0$. We identify this solution with the point $\xi \in \mathbb{R}^n$.

Since delay equations of form (1) are typically more difficult to solve—either analytically or numerically—than ordinary differential equations, it is often advantageous to look for an ordinary differential equation (ODE) that can be used to determine the properties of some of the solutions of the delay equation. In particular, the restriction of a delay equation to a smooth inertial manifold is an ODE. Thus, the physically relevant solutions of a delay equation model with sufficiently small delays can be obtained from an ODE on a finite-dimensional manifold, the restriction of the delay equation to its inertial manifold.

A common expedient in physics is to replace the right side of the delay equation (1) by its expansion as a finite Taylor series to some order N in the delay parameter τ. Since this replacement is an Nth-order ODE, one can hope that some of its solutions might approximate the solutions of the original delay equation (1). For example, for $N = 1$, this procedure yields

$$\dot{x}(t) = f(x(t), x(t)) + \tau D_2 f(x(t), x(t))\dot{x}(t) + O(\tau^2). \tag{2}$$

By ignoring the $O(\tau^2)$ term, we can replace the delay equation with a first-order ODE.

There is a problem with this approach: for $N \geq 2$, the solution space of the order-N approximation to (1) has too large a dimension, namely nN instead of the dimension n of the inertial manifold. A possible way around this difficulty is to restrict the Nth-order approximate ODE to an invariant submanifold of its solutions such that, for each vector $\xi \in \mathbb{R}^n$, there exists a unique solution x (of the ODE) in the submanifold that satisfies the initial condition $x(0) = \xi$. If the delay τ is sufficiently small, then there is such a submanifold, called a *slow* submanifold, which has the expected dimension n of the inertial manifold (see [2]). The slow manifold will be identified with $\mathbb{R}^n$ in the same way as the inertial manifold, via the map $x \to x(0)$.

We can hope that solutions in the slow manifold will approximate, in some appropriate sense, solutions in the inertial manifold. But in what sense?

Once the slow and inertial manifolds are identified with $\mathbb{R}^n$ as described above, solutions in both of these manifolds may be identified with curves in flows defined on $\mathbb{R}^n$. Each of these flows has an infinitesimal generator (a vector field on $\mathbb{R}^n$) that depends on the delay parameter τ. A natural question is whether these infinitesimal generators agree to some order in powers of τ. One of our main results, Theorem 2.8, assures us that they do agree. This theorem is independent of the results in [2]. It states that if sufficiently smooth families of such inertial and slow flows exist, then they agree to order $N - 1$ in τ. This result provides a precise sense in which the slow flow does approximate the inertial flow.

The techniques used in [2], the contraction principle and the fiber contraction principle, produce specific bounds on the range of positive delays τ for which smooth inertial manifolds exist. Here, we are concerned with expansions of vector fields and flows at $\tau = 0$. For this reason, we will require smoothness of families of inertial flows and slow flows in an open neighborhood of $\tau = 0$. The required smoothness is assumed in the main part of the paper, Section 2, on the identity of inertial and slow flows so that this section is self contained; the smoothness of inertial and slow flows is addressed in the latter sections of the paper. The smoothness of inertial flows is proved in Section 4 using the implicit function theorem in an appropriate function space. The existence and smoothness of slow flows is addressed in Section 5 using geometric singular perturbation theory. In addition, a glimpse into the combinatorial-type identities that arise from the identification and agreement of inertial and slow flows is presented in Section 3.

I acknowledge the contributions of Stephen Parrott [12]. Without his help this paper could not have been written.

2 The identity of inertial and slow flows

For a function $g : \mathbb{R}^m \to \mathbb{R}^n$ given by

$$(x_1, x_2, \ldots, x_m) \to g(x_1, x_2, \ldots, x_m),$$

we will use the usual Leibniz notation for partial derivatives, and we will let $D^k g$ denote the kth derivative of g. For a function $g : (\mathbb{R}^j)^m \to (\mathbb{R}^k)^n$ given by

$$(x_1, x_2, \ldots, x_m) \to g(x_1, x_2, \ldots, x_m),$$

the notation $\frac{\partial g}{\partial x_i}(a_1, a_2, \ldots, a_m)$ will be used to denote the derivative of the function

$$x_i \mapsto g(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_m)$$

evaluated at $x_i = a_i$. In addition, $(g)^\ell$ denotes the ℓ-tuple $(g, g, \ldots, g)$

Recall that the derivatives and partial derivatives of a sufficiently smooth function $H : (\mathbb{R}^n)^{N+1} \to \mathbb{R}^n$ given by

$$(x_0, x_1, x_2, \ldots, x_N) \to H(x_0, x_1, x_2, \ldots, x_N), \tag{3}$$

when evaluated at an element of $(\mathbb{R}^n)^{N+1}$, are multilinear forms whose arguments are elements of $(\mathbb{R}^n)^k$, where $k \geq 1$ is the order of the partial derivative or $((R^n)^{N+1})^k$ where k is the order of the derivative. Derivatives and partial derivatives of order zero are defined in the usual manner to be the function H.

Definition 2.1. *Let j be a positive integer and H a function as in display* (3)*. A* differential composition of H *is defined recursively to be a function $(\mathbb{R}^n)^{N+1} \to \mathbb{R}^n$ of one of the following types:*

1. $\zeta \mapsto H(\zeta)$.

2. $\zeta \mapsto \frac{\partial^\beta H}{\partial x_0^{i_0} \partial x_1^{i_1} \cdots \partial x_N^{i_N}}(\zeta)(E_1(\zeta), E_2(\zeta) \dots, E_\beta(\zeta))$, *where* $i_0, i_1, \dots, i_N$ *are nonnegative integers,* $\beta := i_0 + i_2 + \cdots + i_N$*, and* $E_1, E_2 \dots, E_\beta$ *are differential compositions.*

A differential compositional sum (DCS) of H *is a finite linear combination with rational coefficients of differential compositions of* H*.*

We note that H is the unique DCS of H where only zeroth order partial derivatives of H appear. On the other hand, there are an infinite number of DCSs of H where only first-order partial derivatives of H appear. For example,

$$\zeta \mapsto \frac{\partial H}{\partial x_0}(\zeta)(H(\zeta)), \quad \zeta \mapsto \frac{\partial H}{\partial x_0}(\zeta)(\frac{\partial H}{\partial x_0}(\zeta)(H(\zeta))), \dots$$

are all DCSs in H.

In this section, we will consider one-parameter families of flows defined on $\mathbb{R}^n$, where the parameter is defined on an interval of real numbers $J = (-b, b)$ for some $b > 0$ or for $b = \infty$. Each such family is a function $u : \mathbb{R} \times \mathbb{R}^n \times J \to \mathbb{R}^n$ such that $u(0, z, \tau) = z$ and

$$u(s, u(t, z, \tau), \tau) = u(s + t, z, \tau)$$

for all $s, t \in \mathbb{R}$ and $(z, \tau) \in \mathbb{R}^n \times J$.

In all that follows N is a positive integer, $H : (\mathbb{R}^n)^{N+1} \to \mathbb{R}^n$ is a class C^{N+1} function and $Z : \mathbb{R}^n \to (\mathbb{R}^n)^{N+1}$ is given by

$$Z(z) = (z, 0, 0, \dots, 0).$$

We note that a DCS Q of H composed with Z defines a function $\mathbb{R}^n \to \mathbb{R}^n$ given by $z \mapsto Q(Z(z))$. Likewise, for a class C^N family u of flows on $\mathbb{R}^n$, we define the following functions that take values in $(\mathbb{R}^n)^{N+1}$:

$$\begin{aligned}
U &= U(z, \tau) := (u(0, z, \tau), 0, 0, \dots, 0) = (z, 0, 0, \dots, 0), \\
V &= V(z, \tau) := \Big(0, \tau \frac{\partial u}{\partial t}(0, z, \tau), \tau^2 \frac{\partial^2 u}{\partial t^2}(0, z, \tau), \dots, \tau^N \frac{\partial^N u}{\partial t^N}(0, z, \tau)\Big), \\
\tilde{V} &= \tilde{V}(z, \tau) = \Big(0, \frac{\partial u}{\partial t}(0, z, \tau), \tau \frac{\partial^2 u}{\partial t^2}(0, z, \tau), \dots, \tau^{N-1} \frac{\partial^N u}{\partial t^N}(0, z, \tau)\Big). \quad (4)
\end{aligned}$$

Definition 2.2.

1. *For a pair of integers* (j, k) *such that* $1 \leq j \leq N$ *and* $0 \leq k \leq N$, *an indexed set* $P = \{p_i\}_{i=0}^{k}$ *of DCSs, and a class* C^{N+1} *family of flows* $u : \mathbb{R} \times \mathbb{R}^n \times J \to \mathbb{R}^n$ *given by* $(t, z, \tau) \mapsto u(t, z, \tau)$, *the partial derivative* $\partial^j u/\partial t^j$ *is* P-expandable to order k *if there is a subinterval* $K = (-c, c)$ *of* J *with* $c > 0$ *and a class* C^{N+1-j} *function* $R_{j,k} : \mathbb{R}^n \times K \to \mathbb{R}^n$ *such that*
$$\frac{\partial^j u}{\partial t^j}(0, z, \tau) = \sum_{i=0}^{k} \tau^i p_i(Z(z)) + \tau^{k+1} R_{j,k}(z, \tau)$$
for all $(z, \tau) \in \mathbb{R}^n \times K$.

2. *The* C^{2N+1} *family of flows* u *on* $\mathbb{R}^n$ *is* H-admissible *if there is some* $c > 0$ *such that* $K = (-c, c)$ *is a subinterval of* J *and a* C^N *function* $h : \mathbb{R} \times \mathbb{R}^n \times K \to \mathbb{R}^n$ *such that*
$$\begin{aligned}\frac{\partial u}{\partial t}(t, z, \tau) = H(u(t, z, \tau), \tau \frac{\partial u}{\partial t}(t, z, \tau), \ldots, \tau^N \frac{\partial^N u}{\partial t^N}(t, z, \tau))\\ + \tau^{N+1} h(t, z, \tau). \qquad (5)\end{aligned}$$

3. *Given a positive integer* ℓ *and an indexed set* $P = \{p_i\}_{i=0}^{\ell}$ *of DCSs, a class* $C^{\ell+1}$ *family of flows* $u : \mathbb{R} \times \mathbb{R}^n \times J \to \mathbb{R}^n$ *is* P-determined to order ℓ *if* $\partial u/\partial t$ *is* P*-expandable to order* ℓ.

Theorem 2.3. *If* $H : (\mathbb{R}^n)^{N+1} \to \mathbb{R}^n$ *is class* C^{N+1}, *then there is an indexed set* $P = \{p_i\}_{i=0}^{N}$ *of DCSs of* H *such that every* H*-admissible family of flows on* $\mathbb{R}^n$ *is* P*-determined to order* N. *In particular, two* H*-admissible families of flows, parametrized by* τ *in an open interval containing* $\tau = 0$, *have the same degree-*N *Taylor polynomials at* $\tau = 0$.

The proof of Theorem 2.3 will be presented after we discuss its application to the agreement of families of inertial and slow flows for the family of functional differential equations
$$\dot{x}(t) = f(x(t), x(t - \tau)), \qquad (6)$$
parametrized by $\tau \in \mathbb{R}$, where $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is class C^∞.

Definition 2.4. *Suppose that J is an interval of real numbers (including the case where J contains only one point). A family of flows $\eta : \mathbb{R}\times\mathbb{R}^n\times J \to \mathbb{R}^n$ is called* inertial with respect to f *if*

$$\frac{\partial\eta}{\partial t}(t,\xi,\tau) = f(\eta(t,\xi,\tau),\eta(t-\tau,\xi,\tau)) \tag{7}$$

for every $(t,\xi,\tau) \in \mathbb{R}\times\mathbb{R}^n\times J$. The corresponding family of vector fields, given by

$$X(\xi,\tau) := \frac{\partial\eta}{\partial t}(0,\xi,\tau) = f(\xi,\eta(-\tau,\xi,\tau)),$$

is called a family of inertial vector fields.

The following theorem on the existence of smooth families of inertial flows is proved in Section 4.

Theorem 2.5. *Suppose that $f : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ is a C^∞ function and $\sup_{(x,y)\in\mathbb{R}^n} |Df(x,y)| < \infty$. If $b > 0$ is sufficiently small, then there is a C^∞ family of flows $\eta : \mathbb{R}\times\mathbb{R}^n\times(-b,b)\to\mathbb{R}^n$ that is inertial with respect to f.*

Let $y : \mathbb{R}\to\mathbb{R}^n$ be class C^{N+1}. The Nth-order Taylor polynomial of the function $\tau \mapsto f(y(t),y(t-\tau))$ at $\tau = 0$ can be represented in the form

$$T_N(f)(y(t),\tau Dy(t),\tau^2D^2y(t),\dots,\tau^N D^N y(t))$$

where $T_N(f) : (\mathbb{R}^n)^{N+1}\to\mathbb{R}^n$ is a C^∞ function.

Definition 2.6. *For J as in Definition* (2.4) *and an integer $N \geq 0$, a family of flows $\varphi : \mathbb{R}\times\mathbb{R}^n\times J\to\mathbb{R}^n$, given by $(t,\xi,\tau)\mapsto\varphi(t,\xi,\tau)$, is called* order-$N$ slow with respect to f *if*

$$\frac{\partial\varphi}{\partial t}(t,\xi,\tau) = T_N(f)(\varphi(t,\xi,\tau),\tau\frac{\partial\varphi}{\partial t}(t,\xi,\tau),\dots,\tau^N\frac{\partial^N\varphi}{\partial t^N}(t,\xi,\tau)) \tag{8}$$

for every $(t,\xi,\tau)\in\mathbb{R}\times\mathbb{R}^n\times J$. The corresponding family of vector fields, given by

$$Y_N(\xi,\tau) := \frac{\partial\varphi}{\partial t}(t,\xi,\tau))|_{t=0},$$

is called a family of order-N slow vector fields.

Theorem 2.7. *Suppose that* $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is class* C^∞ *and* $N \geq 2$ *is an even integer. If* $D_2 f(x,x)$ *is invertible for all* $x \in \mathbb{R}^n$*, then there is a* C^{N-1} *family* u *of order-*N *slow flows (parametrized by* τ *in an open interval* J *containing the origin) associated with the family of delay equations* (22)*. The family of generators* $\mathcal{S}$ *for* u *is class* C^{N-1}*. Moreover, there is a* $T_N(f)$*-admissible family of flows* φ *such that*

$$\frac{\partial^j \varphi}{\partial \tau^j}(t,\xi,0) = \frac{\partial^j u}{\partial \tau^j}(t,\xi,0)$$

and

$$\frac{\partial^{j+1} \varphi}{\partial t \partial \tau^j}(t,\xi,0) = \frac{\partial^{j+1} u}{\partial t \partial \tau^j}(t,\xi,0)$$

for $(t,\xi) \in \mathbb{R} \times \mathbb{R}^n$ *and* $j = 0,1,2,\ldots,N-1$*.*

The proof of Theorem 2.7 uses singular perturbation theory. A brief discussion of this subject and an outline of the proof of Theorem 2.7 is given in Section 5 (see Theorem 5.1).

We will use Theorem 2.7 together with Theorem 2.3 to prove the next result, a strong confirmation of the conjecture on inertial and slow flows stated in [2].

Theorem 2.8. *Suppose that* $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is a class* C^∞ *function,* $b > 0$*,* J *is the open interval* $(-b,b)$*, and* N *is a positive integer. If* $X : \mathbb{R}^n \times J \to \mathbb{R}^n$ *is a class* C^{2N+1} *family of inertial vector fields with respect to* f *and* $\mathcal{S} : \mathbb{R}^n \times J \to \mathbb{R}^n$ *is a class* C^{N-1} *order-*N *family of slow vector fields with respect to* f*, then there is a number* $c > 0$*, with* $c < b$*, and a continuous function* $R : \mathbb{R}^n \times (-c,c) \to \mathbb{R}^n$ *such that* $X(\xi,\tau) = \mathcal{S}(\xi,\tau) + \tau^{N-1} R(\xi,\tau)$*. Moreover,*

$$\frac{\partial^j X}{\partial \tau^j}(x,0) = \frac{\partial^j \mathcal{S}}{\partial \tau^j}(x,0)$$

for $j = 0,1,2,\ldots,N-1$*.*

Proof. The family of inertial flows $\eta : \mathbb{R} \times \mathbb{R}^n \times J \to \mathbb{R}^n$ corresponding to X is class C^{2N+1}. We claim that there is a number $c > 0$ such that $c < b$, a C^∞ function $H : (\mathbb{R}^n)^{N+1} \to \mathbb{R}^n$ and a C^N function $h : \mathbb{R} \times \mathbb{R}^n \times (-c,c) \to \mathbb{R}^n$ such that

$$\frac{\partial \eta}{\partial t}(t,\xi,\tau) = H(\eta(t,\xi,\tau), \tau \frac{\partial \eta}{\partial t}(t,\xi,\tau), \ldots, \tau^N \frac{\partial^N \eta}{\partial t^N}(t,\xi,\tau)) + \tau^{N+1} h(t,z,\tau).$$

To prove the claim, note first that if $y : \mathbb{R} \to \mathbb{R}^n$ is class C^{2N+1}, then there is an interval $(-c, c)$ and a C^N function $Q : \mathbb{R} \times (-c, c) \to \mathbb{R}^n$ such that

$$\begin{aligned} f(y(t), y(t-\tau)) = {} & T_N(f)(y(t), \tau Dy(t), \tau^2 D^2 y(t), \ldots, \tau^N D^N y(t)) \\ & + \tau^{N+1} Q(t, \tau). \end{aligned}$$

With $y(t) := \eta(t, \xi, \tau)$, we obtain the equation

$$\begin{aligned} \frac{\partial \eta}{\partial t}(t, \xi, \tau) = {} & T_N(f)(\eta(t, \xi, \tau), \tau \frac{\partial \eta}{\partial t}(t, \xi, \tau), \ldots, \tau^N \frac{\partial^N \eta}{\partial t^N}(t, \xi, \tau)) \\ & + \tau^{N+1} P(t, \tau, \xi, \tau), \end{aligned}$$

where P is class C^N. Therefore, the family of flows η is $T_N(f)$-admissible.

Let $\varphi : \mathbb{R} \times \mathbb{R}^n \times J \to \mathbb{R}^n$ denote a $T_N(f)$-admissible family of flows as in Theorem 2.7 and let Z denote its generator. By an application of Theorem 2.3 the families of flows η and ϕ agree to order N in τ.

For a class C^r family of flows ψ on $\mathbb{R}^n$ with generator W, we have by definition

$$\frac{\partial \psi}{\partial t}(t, \xi, \tau) = W(\psi(t, \xi, \tau)). \tag{9}$$

Since,

$$W(\xi, \tau) = \frac{\partial \psi}{\partial t}(0, \xi, \tau),$$

the vector field W is class C^{r-1}. By differentiating in equation (9) and using an induction argument, we have that

$$\frac{\partial^{j+1} \psi}{\partial t \partial \tau^j}(0, x, 0) = \frac{\partial^j W}{\partial \tau^j}(x, 0) \tag{10}$$

for $j = 0, 1, 2, \ldots, r-1$. Using this fact, it follows that if η and ϕ agree to order N in τ, then there is a continuous function R_1 such that

$$X(\xi, \tau) - Z(\xi, \tau) = \tau^N R_1(\xi, \tau).$$

Since φ and u agree to order $N-1$, the generator $\mathcal{S}$ for u is class C^{N-1} (by Theorem 2.7), and in view of the result expressed by equation (10), there is a continuous function R_2 such that

$$Z(\xi, \tau) - \mathcal{S}(\xi, \tau) = \tau^{N-1} R_2(\xi, \tau).$$

Hence

$$\begin{aligned} X(\xi,\tau) - \mathcal{S}(\xi,\tau) &= (X(\xi,\tau) - Z(\xi,\tau)) + (Z(\xi,\tau) - \mathcal{S}(\xi,\tau)) \\ &= \tau^N R_1(\xi,\tau) - \tau^{N-1} R_2(\xi,\tau) \\ &= \tau^{N-1}(\tau R_1(\xi,\tau) - R_2(\xi,\tau)), \end{aligned}$$

as required. □

The remainder of this section contains a proof of Theorem 2.3.

Lemma 2.9. *Suppose that k is an integer, with $0 \le k \le N-1$, the function $H : (\mathbb{R}^n)^{N+1} \to \mathbb{R}^n$ is class C^{N+1}, and $P^1 = \{p_{1,i}\}_{i=0}^k$ is an indexed set of DCSs of H. If every H-admissible family of flows u is P^1-determined to order k, then there are indexed sets*

$$P^2 = \{p_{2,i}\}_{i=0}^{k-1},\ P^3 = \{p_{3,i}\}_{i=0}^{k-2},\ \ldots,\ P^{k+1} = \{p_{k+1,i}\}_{i=0}^{0}$$

of DCSs of H such that for every H-admissible family of flows u and integer $j \in \{2, 3, \ldots, k+1\}$, the partial derivative $\partial^j u/\partial t^j$ is P^j-expandable to order $k - j + 1$.

Proof. Let u be H-admissible. According to the hypothesis of the lemma, $\partial u/\partial t$ is P^1-expandable to order k; that is,

$$\frac{\partial u}{\partial t}(0, z, \tau) = \sum_{i=0}^{k} \tau^i p_{1,i}(Z(z)) + \tau^{k+1} R_{1,k}(z,\tau),$$

where $R_{1,k}$ is class C^N.

Induction Hypothesis: For j an integer, with $0 \le j \le k$, there are indexed sets

$$P^1 = \{p_{1,i}\}_{i=0}^{k},\ P^2 = \{p_{2,i}\}_{i=0}^{k-1},\ \ldots,\ P^{j+1} = \{p_{j+1,i}\}_{i=0}^{k-j}$$

of DCSs of H such that, for every H-admissible family of flows u on $\mathbb{R}^n$ and integer $\ell = 1, 2, \ldots, j+1$, the partial derivative $\partial^\ell u/\partial t^\ell$ is P^ℓ-expandable to order $k - \ell + 1$.

The induction hypothesis is true for $j = 0$ by the hypothesis of the lemma. Suppose it is true for some j, with $1 \le j \le k-1$. We will prove that it is true for $j+1$.

By the induction hypothesis, if u is an H-admissible family of flows, then

$$\frac{\partial^j u}{\partial t^j}(0,z,\tau) = \sum_{i=0}^{k-(j-1)} \tau^i p_{j,i}(Z(z)) + \tau^{k-j+2} R_{j,k}(z,\tau), \tag{11}$$

where $R_{j,k}$ is class C^{N+1-j}.

Note that

$$\begin{aligned} \frac{\partial^j u}{\partial t^j}(0,u(s,z,\tau),\tau) &= \frac{d^j}{dt^j} u(t,u(s,z,\tau),\tau)\Big|_{t=0} \\ &= \frac{d^j}{dt^j} u(t+s,z,\tau)\Big|_{t=0} \\ &= \frac{\partial^j u}{\partial t^j}(s,z,\tau); \end{aligned}$$

therefore, after replacing s by t, we have

$$\frac{\partial^j u}{\partial t^j}(0,u(t,z,\tau),\tau) = \frac{\partial^j u}{\partial t^j}(t,z,\tau).$$

After replacing $z \in \mathbb{R}^n$ by $u(t,z,\tau) \in \mathbb{R}^n$ in equation (11), it follows that

$$\frac{\partial^j u}{\partial t^j}(t,z,\tau) = \sum_{i=0}^{k-(j-1)} \tau^i p_{j,i}(Z(u(t,z,\tau))) + \tau^{k-j+2} R_{j,k}(u(t,z,\tau),\tau). \tag{12}$$

By differentiation of both sides of equation (12) with respect to t at $t = 0$, we have that

$$\begin{aligned} \frac{\partial^{j+1} u}{\partial t^{j+1}}(0,z,\tau) = \sum_{i=0}^{k-(j-1)} & \tau^i Dp_{j,i}(Z(z)) Z(\frac{\partial u}{\partial t}(0,z,\tau)) \\ & + \tau^{k-j+2} D_1 R_{j,k}(z,\tau) \frac{\partial u}{\partial t}(0,z,\tau). \end{aligned} \tag{13}$$

Note that each summand in the first sum in equation (13) is a DCS of H. By the induction hypothesis, $\partial u/\partial t$ is P^1-expandable to order k. By substitution of this expansion for $\partial u/\partial t$ in the last equation, we find that $\partial^{j+1} u/\partial t^{j+1}$ is P^{j+1}-expandable to order $k-j+1$ for an indexed set $P^{j+1} = \{p_{j+1,i}\}_{i=0}^{k-j}$ of DCSs of H that does not depend on the choice of u. This set P^{j+1} is formed from the coefficients of the rearrangement of the equation (13) into a series in powers of τ up to order k. □

Proof of Theorem 2.3. The proof is by induction on the order of determinacy. **Induction Hypothesis:** For an integer k, with $0 \le k \le N$, there is a set $P^1(k)$ of DCSs of H such that every H-admissible function is $P^1(k)$-determined to order k.

For each vector $z \in \mathbb{R}^n$ and integer k (with $0 \le k \le N$), the kth-order Taylor expansion of H at $Z := Z(z)$ is given by

$$H(X+Z) = H(Z) + \rho_0(X,Z)X$$

for $k=0$ and

$$H(X+Z) = H(Z) + \sum_{i=1}^{k} \frac{1}{i!} D^i H(Z) X^i + \rho_k(X,Z)X^{k+1} \tag{14}$$

for $k \ge 1$, where ρ_k is a class C^{N-k} function defined for $X = (x_0, x_1, \ldots, x_N)$ in $(\mathbb{R}^n)^{N+1}$ in an open ball $B(z)$ centered at $Z(z)$. In fact,

$$\rho_0(X,Z) = \int_0^1 DH(Z+tX)\,dt$$

and, for $k \ge 1$,

$$\rho_k(X,Z) = \frac{1}{k!}\int_0^1 (1-t)^k D^{k+1} H(Z+tX)\,dt.$$

To obtain this result, we apply Taylor's theorem for the Taylor polynomial of degree $k+1$ (see [1, p. 93]), and then we absorb the degree $k+1$ term of the Taylor polynomial into the remainder.

In case $k=0$, consider an H-admissible family of flows u and choose τ sufficiently small so that the image of V (defined in display (4)) is in B. Also, take $P^1(0) = \{H\}$, an indexed set of DCSs of H of length one. Using equation (5) and equation (14) with $k=0$, we have that

$$\begin{aligned}\frac{\partial u}{\partial t}(0,z,\tau) &= H(Z+V) + \tau^{N+1} h(0,z,\tau)\\ &= H(Z) + \tau R_{1,0}(z,\tau),\end{aligned} \tag{15}$$

where

$$R_{1,0}(z,\tau) := \rho_0(V,Z)\tilde{V} + \tau^N h(0,z,\tau)$$

is class C^N. Hence, $\partial u/\partial t$ is $P^1(0)$-expandable to order zero; that is, u is $P^1(0)$-determined to order zero.

While it is not logically necessary, we will establish the first induction step (that is, the induction hypothesis holds for $k = 1$ if it holds for $k = 0$) to show the idea of the proof in the simplest case. For this, we set

$$P^1(1) = \{H, \frac{\partial H}{\partial x_1} H\},$$

an indexed set of DCSs of H of length two. Also, we note from equation (14) that the first-order Taylor expansion of H at Z is given by

$$H(X + Z) = H(Z) + \sum_{i=0}^{N} \frac{\partial H}{\partial x_i}(Z) x_i + \rho_1(X, Z) X^2,$$

where $X = (x_0, x_1, \dots, x_N)$ is an element of $(\mathbb{R}^n)^{N+1}$. Using equation (5) and replacing X with V as in display (4), it follows that

$$\begin{aligned}
\frac{\partial u}{\partial t}(0, z, \tau) &= H(Z) + \sum_{i=1}^{N} \frac{\partial H}{\partial x_i}(Z)(\tau^i \frac{\partial^i u}{\partial t^i}(0, z, \tau)) \\
&\quad + \rho_1(V, Z)V^2 + \tau^{N+1} h(0, z, \tau) \\
&= H(Z) + \tau \frac{\partial H}{\partial x_1}(Z) \frac{\partial u}{\partial t}(0, z, \tau) \\
&\quad + \tau^2 \big(\sum_{i=2}^{N} \frac{\partial H}{\partial x_i}(Z)(\tau^{i-2} \frac{\partial^i u}{\partial t^i}(0, z, \tau)) + \rho_1(V, Z)(\tilde{V})^2 \\
&\quad + \tau^{N-1} h(0, z, \tau)\big).
\end{aligned}$$

By substituting for $\partial u/\partial t(0, z, \tau)$ on the right-hand side of the last equation using formula (15), we find that

$$\frac{\partial u}{\partial t}(0, z, \tau) = H(Z) + \tau \frac{\partial H}{\partial x_1}(Z) H(Z) + \tau^2 R_{1,1}(z, \tau),$$

where $R_{1,1}$ (defined by collecting the terms of order at least two in τ) is class C^N. Thus, we have proved that $\partial u/\partial t$ is $P^1(1)$-expandable to order one; that is every H-admissible family of flows is $P^1(1)$-determined to order one.

We note that, by Lemma 2.9, there are indexed sets $P^1(1)$ (already defined) and

$$P^2(1) := \{\frac{\partial H}{\partial x_0}(Z(z) H(z)\}$$

of DCSs of H such that $\partial u/\partial t$ is $P^1(1)$-expandable to order one and $\partial^2 u/\partial t^2$ is $P^2(1)$-expandable to order zero. This result would be preparatory for proving the case $k = 2$.

For the general induction step, suppose that the induction hypothesis is true for $k-1$, and apply Lemma 2.9 to obtain indexed sets

$$P^1(k-1) = \{p_{1,i}\}_{i=0}^{k-1},\ P^2(k-1) = \{p_{2,i}\}_{i=0}^{k-2},\ \ldots,\ P^k(k-1) = \{p_{k,i}\}_{i=0}^{0} \quad (16)$$

of DCSs of H such that, for every H-admissible family of flows u and for every $j \in \{1, 2, \ldots, k\}$, the partial derivative $\partial^j u/\partial t^j$ is $P^j(k-1)$-expandable to order $k - j$.

For an H-admissible function u, we have that

$$\frac{\partial u}{\partial t}(0, z, \tau) = H(V + Z) + \tau^{N+1} h(0, z, \tau).$$

Using the kth order Taylor expansion of H, this equation can be replaced by

$$\begin{aligned}\frac{\partial u}{\partial t}(0, z, \tau) = H(Z) &+ \sum_{i=1}^{k} \frac{\tau^i}{i!} D^i H(Z) V^i \\ &+ \tau^{k+1}(\rho_k(V, Z)\tilde{V}^{k+1} + \tau^{N-k} h(0, z, \tau)). \qquad (17)\end{aligned}$$

We use the result of applying Lemma 2.9 to replace the last N components of $\tilde{V}$ by their corresponding expansions (with remainders) relative to the indexed sets of DCSs of H mentioned in display (16). There are two types of terms in these expansions: $\tau^q P_{j,\ell}(Z(z))$ and $\tau^r R_{j,k-1}(z, \tau)$, where $P_{j,\ell}$ is a DCS of H, $0 \le q \le k - 1$, and $r \ge k$.

Recall that $D^i H(Z(z))(\,\cdot\,, \cdot, \ldots, \cdot)$ is a multi-linear function with i arguments and use Lemma 2.9 to replace the last N components of V by their corresponding expansions (with remainders) relative to the indexed sets of DCSs of H mentioned in display (16) as follows:

$$\begin{aligned} V &:= \Big(0,\ \tau \frac{\partial u}{\partial t}(0, z, \tau)\,,\ \tau^2 \frac{\partial^2 u}{\partial t^2}(0, z, \tau)\,, \ldots,\ \tau^N \frac{\partial^N u}{\partial t^N}(0, z, \tau)\Big) \\ &= \Big(0,\ \tau\Big[\sum_{i=0}^{k-1} p_{1,i}\tau^i + \tau^k R_{1,k-1}\Big],\ \tau^2\Big[\sum_{i=0}^{k-2} p_{2,i}\tau^i + \tau^{k-1} R_{2,k-1}\Big]\,, \ldots, \\ &\qquad \tau^k\Big[\sum_{i=0}^{0} p_{k,i}\tau^i + \tau R_{k,k-1}\Big]\,,\ O(\tau^{k+1})\,, \ldots,\ O(\tau^N)\,\Big). \end{aligned}$$

Insert the last expression for V as arguments in the multilinear function $D^i H(Z)$ in (17), expand using multilinearity, and collect terms according to powers of τ.

We are only interested in the form of the terms corresponding to powers of τ no greater than k. All other terms are collected into a $O(\tau^{k+1})$ grand remainder term. In particular, all terms arising from the $R_{i,j}$ in the components of V are collected into the grand remainder. The only terms *not* collected into the grand remainder arise from the $p_{j,i}$ in the components of V. These $p_{j,i}$, which were furnished by the induction hypothesis and Lemma 2.9, are *independent of the H-admissible flow.*

The terms arising from $p_{j,i}$, when collected according to the powers of τ with which they are associated, form the desired indexed set $\mathcal{P}^1(k)$ of DCSs of H. The set $\mathcal{P}^1(k)$ has the desired property that every H-admissible family of flows is $\mathcal{P}^1(k)$-determined. This shows that if the induction hypothesis is true for $k-1$, then it is true for k. $\square$

3 Curious Combinatorial-style Identities

A byproduct of Theorem 2.8 is a class of combinatorial-style identities such as

$$\begin{aligned}
&2^{m-1} + m(m-1)2^{m-2} - (2+m)^{m-1} \\
&= \sum_{j=0}^{m-2} \sum_{\ell=0}^{j} \Bigg(\binom{j}{\ell} \binom{m}{j} (j-\ell)(m-\ell)^{m-j-1} \\
&\quad + \sum_{\alpha=1}^{m-j-1} \binom{j}{\ell} \binom{m}{j} \binom{m-j}{\alpha} \ell(\ell+\alpha)^{\alpha-1}(j-\ell)(m-\ell)^{m-j-\alpha-1} \\
&\quad + \binom{j}{\ell} \binom{m}{j} \ell(\ell+m-j)^{m-j-1} \Bigg),
\end{aligned}$$

which holds for $m \geq 2$, and

$$\sum \ell!\beta! \binom{n}{\beta} \frac{2^{b_1} 3^{2b_2} \cdots (\ell+1)^{\ell b_\ell}}{b_1! b_2! \cdots b_\ell! (1!2)^{b_1} (2!3)^{b_2} \cdots (\ell!(\ell+1))^{b_\ell}} = n(n+\ell)^{\ell-1},$$

where $n \geq 1$, $\ell \geq 1$, and the sum is over all ℓ-tuples of nonnegative integers $(b_1, b_2, \ldots, b_\ell)$ such that $\beta := b_1 + b_2 + \cdots + b_\ell$ and $b_1 + 2b_2 + \cdots + \ell b_\ell = \ell$ (cf. Faà di Bruno's formula [6, 8]).

Many other identities can be obtained by computing in various ways and equating the power series expansions for inertial and slow vector fields corresponding to delay equations that can be solved explicitly. The identities given in this section are derived from the scalar delay equation

$$\dot{x}(t) = ax(t-\tau) + bx(t).$$

Taken out of context, these identities would perhaps pose a challenge to verify; for example, they do not seem to be provable by the available computer algorithms [10]. In our context, their verification is transparent.

It is not clear if identities of this type have applications to combinatorics, but they seem sufficiently unusual to be worth mentioning.

4 Inertial Flows

Theorem 4.1. *Suppose that* $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is a* C^∞ *function and* $\sup_{(x,y)\in\mathbb{R}^n} |Df(x,y)| < \infty$. *If* $b > 0$ *is sufficiently small, then there is a* C^∞ *family of flows* $\eta : \mathbb{R} \times \mathbb{R}^n \times (-b, b) \to \mathbb{R}^n$ *that is inertial with respect to* f.

Proof. We will prove there is a C^1 family of flows. The existence of higher-order derivatives can be proved by induction.

It suffices to show that there is an interval $(-b, b)$ and a smooth function $u : \mathbb{R} \times \mathbb{R}^n \times (-b, b) \to \mathbb{R}^n$ such that for each $\xi \in \mathbb{R}^n$ and $\tau \in (-b, b)$ the function $t \mapsto u(t, \xi, \tau)$ is the unique solution of the functional differential equation $\dot{x}(t) = f(x(t), x(t-\tau))$ such that $u(0, \xi, \tau) = \xi$. In this case, the function u is a flow. In fact $t \mapsto u(t+s, \xi, \tau)$ and $t \mapsto u(t, u(s, \xi, \tau), \tau)$ are both solutions with the same value at $t = 0$. Hence, by the uniqueness, they are equal.

The idea of the proof is to change coordinates and use the implicit function theorem (cf. [11]).

For $\delta > 0$, define $\sigma := t/\delta$, $\rho = \tau/\delta$, and $z(\sigma, \xi, \rho) := u(\delta\sigma, \xi, \delta\rho) - \xi$. It suffices to prove the existence, uniqueness, and smooth dependence on parameters for the solution $\sigma \mapsto z(\sigma, \xi, \rho)$ of the initial value problem

$$\dot{v}(\sigma) = \delta f(\xi + v(\sigma), \xi + v(\sigma - \rho)), \qquad v(0) = 0, \tag{18}$$

where δ is some positive real number.

Suppose that $\gamma : \mathbb{R} \to \mathbb{R}$ is differentiable and $(\delta, \xi, r) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$. We will consider the operator

$$F(\delta, \xi, \rho, \gamma)(\sigma) := \gamma'(\sigma) - \delta f(\xi + \gamma(\sigma), \xi + \gamma(\sigma - \rho)). \tag{19}$$

Solutions of the initial value problem (18) correspond to zeros of F.

To apply the implicit function theorem to F, we will require γ to belong to an appropriate function space of continuous, exponentially bounded functions. For this purpose, we define $\mathrm{CEB}^{r,\lambda_0,\lambda_1,\ldots,\lambda_r}(\mathbb{R}, \mathbb{R}^n)$ to be the set of all C^r-functions $\gamma : \mathbb{R} \to \mathbb{R}^n$ such that

$$\|\gamma\|_{r,\lambda_0,\lambda_1,\ldots,\lambda_r} := \sup_{\sigma \in \mathbb{R}} e^{-\lambda_0|\sigma|}|\gamma(\sigma)| + e^{-\lambda_1|\sigma|}|\gamma'(\sigma)| + \cdots + e^{-\lambda_r|\sigma|}|\gamma^{(r)}(\sigma)| < \infty.$$

Suppose that r is a nonnegative integer and $\lambda_0, \lambda_1, \ldots, \lambda_r$ are nonnegative real numbers. The set $\mathrm{CEB}^{r,\lambda_0,\lambda_1,\ldots,\lambda_r}(\mathbb{R}, \mathbb{R}^n)$ with the norm $\|\|_{r,\lambda_0,\lambda_1,\ldots,\lambda_r}$ is a Banach space (cf. [3, Appendix IV]). Moreover, the subset $\mathrm{CEB}_0^{r,\lambda_0,\lambda_1,\ldots,\lambda_r}(\mathbb{R}, \mathbb{R}^n)$ consisting of all functions in $\mathrm{CEB}^{r,\lambda_0,\lambda_1,\ldots,\lambda_r}(\mathbb{R}, \mathbb{R}^n)$ that vanish at the origin is a closed subspace.

Claim 4.2. *If $\lambda > 0$, then the operator d defined by $d\gamma = \gamma'$ is an isomorphism $d : \mathrm{CEB}_0^{1,\lambda,\lambda}(\mathbb{R}, \mathbb{R}^n) \to \mathrm{CEB}^{0,\lambda}(\mathbb{R}, \mathbb{R}^n)$.*

To prove Claim 4.2, let us suppose that $\gamma \in \mathrm{CEB}_0^{1,\lambda,\lambda}(\mathbb{R}, \mathbb{R}^n)$. Using the estimate

$$e^{-\lambda|\sigma|}|\gamma'(s)| \leq \|\gamma\|_{1,\lambda,\lambda}, \tag{20}$$

it follows that $\|\gamma'\|_{0,\lambda} \leq \|\gamma\|_{1,\lambda,\lambda}$. Hence, $\gamma' \in \mathrm{CEB}^{0,\lambda}(\mathbb{R}, \mathbb{R}^n)$; that is, d is a linear operator from $\mathrm{CEB}_0^{1,\lambda,\lambda}(\mathbb{R}, \mathbb{R}^n)$ to $\mathrm{CEB}^{0,\lambda}(\mathbb{R}, \mathbb{R}^n)$.

To finish the proof of Claim 4.2, we will show that the operator L defined by

$$(L\gamma)(\sigma) = \int_0^\sigma \gamma(s)\, ds$$

is a bounded (linear) operator $L : \mathrm{CEB}^{0,\lambda}(\mathbb{R}, \mathbb{R}^n) \to \mathrm{CEB}_0^{1,\lambda,\lambda}(\mathbb{R}, \mathbb{R}^n)$ and L is the inverse of d.

If γ is continuous, then $L\gamma$ is class C^1. Also, $(L\gamma)(0) = 0$. We will show that

$$\|L\gamma\|_{1,\lambda,\lambda} \leq \frac{\lambda + 2}{\lambda}\|\gamma\|_{0,\lambda}; \tag{21}$$

that is $L : \mathrm{CEB}^{0,\lambda}(\mathbb{R}, \mathbb{R}^n) \to \mathrm{CEB}_0^{1,\lambda,\lambda}(\mathbb{R}, \mathbb{R}^n)$ is bounded. In fact, we have that

$$\begin{aligned}
e^{-\lambda|\sigma|}|(L\gamma)(\sigma)| + e^{-\lambda|\sigma|}|(L\gamma)'(\sigma)| &= e^{-\lambda|\sigma|}|\int_0^\sigma \gamma(s)\,ds| + e^{-\lambda|\sigma|}|\gamma(\sigma)| \\
&= e^{-\lambda|\sigma|}|\int_0^s e^{\lambda|s|}\|\gamma\|_{0,\lambda}\,ds| + \|\gamma\|_{0,\lambda} \\
&= \|\gamma\|_{0,\lambda} e^{-\lambda|\sigma|}(\frac{1}{\lambda}(1 + e^{\lambda|\sigma|}) + \|\gamma\|_{0,\lambda} \\
&= \frac{\lambda + 2}{\lambda}\|\gamma\|_{0,\lambda}.
\end{aligned}$$

This proves the norm estimate (21).

Finally, it is easy to see that Ld and dL are both the identity. This proves Claim 4.2.

Claim 4.3. *If $\lambda > 0$, then the operator F defined in display* (19) *defines a class C^1 function*

$$F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathrm{CEB}_0^{1,\lambda,\lambda}(\mathbb{R}, \mathbb{R}^n) \to \mathrm{CEB}^{0,\lambda}(\mathbb{R}, \mathbb{R}^n).$$

To prove Claim 4.3, we show first that F is defined as indicated. Suppose that $(\delta, \xi, \rho) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ and $\gamma \in \mathrm{CEB}_0^{1,\lambda,\lambda}(\mathbb{R}, \mathbb{R}^n)$. In view of Claim 4.2, it suffices to consider only the second summand in the definition of F. Clearly, $\sigma \mapsto \delta f(\xi + \gamma(\sigma), \xi + \gamma(\sigma - \rho))$ is C^1 (in particular, class C^0). Using the mean value theorem and the boundedness of the derivative of f in the hypothesis, we have that

$$\begin{aligned}
e^{-\lambda|\sigma|}|\delta f(\xi + \gamma(\sigma), \xi + \gamma(\sigma - \rho))| &\le \delta e^{-\lambda|\sigma|}(|f(\xi, \xi)| \\
&\quad + |f(\xi + \gamma(\sigma), \xi + \gamma(\sigma - \rho)) - f(\xi, \xi)|) \\
&\le \delta e^{-\lambda|\sigma|}\|Df\|(|f(\xi, \xi)| \\
&\quad + |\gamma(\sigma)| + |\gamma(\sigma - \rho)|) \\
&\le \delta\|Df\|(|f(\xi, \xi)| + \|\gamma\|_{1,\lambda,\lambda} \\
&\quad + e^{-\lambda|\sigma|}e^{\lambda|\sigma-\rho|}\|\gamma\|_{1,\lambda,\lambda}) \\
&\le \delta\|Df\|(|f(\xi, \xi)| + \|\gamma\|_{1,\lambda,\lambda} \\
&\quad + e^{\lambda|\rho|}\|\gamma\|_{1,\lambda,\lambda}) < \infty;
\end{aligned}$$

that is, the range of F is in $\mathrm{CEB}^{0,\lambda}(\mathbb{R}, \mathbb{R}^n)$.

It remains to show that F is class C^1. The first term in the formula (19) is linear, hence class C^∞. Thus, it suffices to prove that the function defined by the second term is C^1. The essential step is to show that the operator on $\mathrm{CEB}_0^{1,\lambda,\lambda}(\mathbb{R},\mathbb{R}^n)$ given by composition on the left by a C^1 function is C^1. Hence, for simplicity, let us assume that $g : \mathbb{R}^n \to \mathbb{R}^n$ is class C^1 and Dg is bounded. We will show that the function $G : \mathrm{CEB}_0^{1,\lambda,\lambda}(\mathbb{R},\mathbb{R}^n) \to \mathrm{CEB}^{0,\lambda}(\mathbb{R},\mathbb{R}^n)$, given by $\gamma \mapsto g \circ \gamma$, is C^1. The proof has two main steps: We will show that

$$DG : \mathrm{CEB}_0^{1,\lambda,\lambda}(\mathbb{R},\mathbb{R}^n) \to \mathcal{L}(\mathrm{CEB}_0^{1,\lambda,\lambda}(\mathbb{R},\mathbb{R}^n), \mathrm{CEB}^{1,\lambda}(\mathbb{R},\mathbb{R}^n))$$

is given by

$$(DG(\gamma)\nu)(s) = Dg(\gamma(s))\nu(s),$$

and the map DG is continuous.

For the first step, let

$$Q := e^{-\lambda|s|}|g(\gamma(s)+\nu(s)) - g(\gamma(s)) - Dg(\gamma(s))(\nu(s))|.$$

We have the estimates

$$\begin{aligned} Q &\le e^{-\lambda|s|}\Big|\int_0^1 (Dg(\gamma(s)+t\nu(s)) - Dg(\gamma(s))\nu(s)\,dt\Big| \\ &\le e^{-\lambda|s|}|\nu(s)| \sup_{t\in[0,1]} |Dg(\gamma(s)+t\nu(s)) - Dg(\gamma(s))| \\ &\le \|\nu(s)\|_{1,\lambda,\lambda} \sup_{t\in[0,1]} |Dg(\gamma(s)+t\nu(s)) - Dg(\gamma(s))| \end{aligned}$$

Hence,

$$\frac{1}{\|\nu(s)\|_{1,\lambda,\lambda}}\|G(\gamma+\nu) - G(\gamma) - (DG\circ\gamma)\nu\|_{0,\lambda} \le \sup_{s\in\mathbb{R}} \sup_{t\in[0,1]} e^{-\lambda|s|}|Dg(\gamma(s)+t\nu(s)) - Dg(\gamma(s))|.$$

By the definition of differentiation, it suffices to show that the right hand side of the last inequality converges to zero as $\|\nu(s)\|_{1,\lambda,\lambda}$ goes to zero.

Let $\epsilon > 0$ be given. Since Dg is bounded, there is a number $a > 0$ such that

$$2e^{-\lambda a}\|Dg\|_0 < \epsilon,$$

where the norm is the usual supremum norm. Also, since $\gamma([-a,a])$ is a compact subset of $\mathbb{R}^n$, there is a number $b>0$ such that $\|\gamma\|_{1,\lambda,\lambda} < (b/2)e^{-\lambda a}$ and $\gamma([-a,a])$ is contained in the ball B of radius b centered at the origin of $\mathbb{R}^n$.

Since Dg is continuous, it is uniformly continuous on B Hence, there is a number $\delta > 0$ such that $|Dg(z) - Dg(w)| < \epsilon$ whenever $z, w \in B$ and $|z-w| < \delta$. If $s \in [-a,a]$ and $\|\nu\|_{1,\lambda,\lambda} < (b/2)e^{-\lambda a}$, then $\gamma(s)$ and $\gamma(s)+t\nu(s)$ are both in B. If in addition, $\|\nu\|_{1,\lambda,\lambda} < \delta e^{-\lambda a}$, then

$$\frac{1}{\|\nu(s)\|_{1,\lambda,\lambda}}\|G(\gamma+\nu) - G(\gamma) - (DG\circ\gamma)\nu\|_{0,\lambda} < \epsilon,$$

as required.

To prove the DG is continuous, choose $\gamma \in \mathrm{CEB}_0^{1,\lambda,\lambda}(\mathbb{R},\mathbb{R}^n)$. It suffices to show that DG is continuous at γ. Thus, it suffices to estimate the quantity

$$e^{-\lambda|s|}|Dg(\kappa(s))\nu(s) - Dg(\gamma(s))\nu(s)|$$

for $\kappa, \nu \in \mathrm{CEB}_0^{1,\lambda,\lambda}(\mathbb{R},\mathbb{R}^n)$. The required inequalities have already been obtained in the last two paragraphs.

Let us note that $F(0,\xi,0,0) = 0$ and $F_\gamma(0,\xi,0,0) = d$, where d is the operator defined in Claim 4.2. The derivative is easy to compute: we simply set $\delta = 0$ and note that in this case $F(0,\xi,0,\gamma)$ is a linear function of γ. By Claim 4.2, the operator d is invertible. Hence, by an application of the implicit function theorem, there are open neighborhoods U of the origin in $\mathbb{R}$, V of $\xi \in \mathbb{R}^n$, W of the origin in $\mathbb{R}$, and $\mathcal{B}$ of the origin in $\mathrm{CEB}_0^{1,\lambda,\lambda}$, and a C^1 function $\psi : U \times V \times W \to \mathcal{B}$ such that $\psi(0,\xi,0) = 0$ and $F(\delta,\xi,\rho,\psi(\delta,\xi,\rho)) \equiv 0$. Moreover, if $F(\delta,\xi,\rho,\gamma) = 0$ and $(\delta,\xi,\rho,\gamma) \in U \times V \times W \times \mathcal{B}$, then $\psi(\delta,\xi,\rho) = \gamma$.

By choosing $\delta > 0$ in U, we have proved the existence of a unique solution of the initial value problem (18) that depends smoothly on $\xi \in \mathbb{R}^n$ and $\rho \in W$. In particular, the solution is smooth at $\rho = 0$. $\square$

5 Slow Flows

We will discuss the properties, especially the smoothness at $\tau = 0$, of the slow flows associated with the family of delay equations

$$\dot{x} = f(x(t), x(t-\tau)), \tag{22}$$

where the function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is class C^∞ and τ is the parameter.

Let $T_N(f)$ denote the Nth-order Taylor polynomial of the function $\tau \mapsto f(x(t), x(t-\tau))$ and recall that an order-N family of slow flows associated with the delay equation (22) is a family of flows u on $\mathbb{R}^n$, parametrized by τ, such that

$$\frac{\partial u}{\partial t}(t,\xi,\tau) = T_N(f)(u(t,\xi,\tau), \tau\frac{\partial u}{\partial t}(t,\xi,\tau), \ldots, \tau^N \frac{\partial^N u}{\partial t^N}(t,\xi,\tau))$$

for every $(t,\xi,\tau) \in \mathbb{R} \times \mathbb{R}^n \times J$. In other words, an order-N family of slow flows is a family of flows on $\mathbb{R}^n$ that happens to solve the family of ODEs

$$x^{(1)} = T_N(f)(x, \tau^1 x^{(1)}, \tau^2 x^{(2)}, \ldots, \tau^N x^{(N)}). \tag{23}$$

Also, a C^{2N+1} family of flows φ on $\mathbb{R}^n$ is $T_N(f)$-admissible if there is some $c > 0$ such that $K = (-c, c)$ is a subinterval of J and a C^N function $h : \mathbb{R} \times \mathbb{R}^n \times K \to \mathbb{R}^n$ such that

$$\begin{aligned}\frac{\partial \varphi}{\partial t}(t,z,\tau) = T_N(f)(\varphi(t,z,\tau), \tau\frac{\partial \varphi}{\partial t}(t,z,\tau), \ldots, \tau^N \frac{\partial^N \varphi}{\partial t^N}(t,z,\tau)) \\ + \tau^{N+1} h(t,z,\tau).\end{aligned} \tag{24}$$

Theorem 5.1. *Suppose that $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is class C^∞ and $N \geq 2$ is an even integer. If $D_2 f(x,x)$ is invertible for all $x \in \mathbb{R}^n$, then there is a C^{N-1} family u of order-N slow flows (parametrized by τ in an open interval J containing the origin) associated with the family of delay equations* (22). *The family of generators $\mathcal{S}$ for u is class C^{N-1}. Moreover, there is a $T_N(f)$-admissible family of flows φ such that*

$$\frac{\partial^j \varphi}{\partial \tau^j}(t,\xi,0) = \frac{\partial^j u}{\partial \tau^j}(t,\xi,0)$$

and

$$\frac{\partial^{j+1} \varphi}{\partial t \partial \tau^j}(t,\xi,0) = \frac{\partial^{j+1} u}{\partial t \partial \tau^j}(t,\xi,0)$$

for $(t,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and $j = 0, 1, 2, \ldots, N-1$.

The remainder of this section is an outline of the ideas that can be used to construct a proof of Theorem 5.1.

We will use geometric singular perturbation theory in the form discussed by C.K.R.T. Jones [9]. This subject was introduced by N. Fenichel [5] as an application of the persistence results for normally hyperbolic invariant manifolds (see [4, 7]). To apply this theory, we will construct an appropriate first-order singular system that gives rise to a family of slow flows corresponding to the delay equation $\dot{x}(t) = f(x(t), x(t-\tau))$ (cf. [2]). To obtain sufficient smoothness, we will take full advantage of the hypothesis that f is class C^∞.

Note that τ is restricted to $\tau \geq 0$ in the delay equation (22), but the family of differential equations (23) is defined for $\tau \in \mathbb{R}$.

Let $\ell \geq 2$ be an even integer and consider the family of ODEs

$$x^{(1)} = T_\ell(f)(x, \tau^1 x^{(1)}, \tau^2 x^{(2)}, \ldots, \tau^\ell x^{(\ell)}). \tag{25}$$

For our purposes, the appropriate first-order system equivalent to the family of ODEs (25) is parametrized by $\mu := \tau^{1/(\ell-1)}$ and given by

$$\begin{aligned} \dot{x} &= y_1, \\ \mu^\ell \dot{y}_1 &= y_2, \\ &\vdots \\ \mu^\ell \dot{y}_{\ell-2} &= y_{\ell-1}, \\ \mu^\ell (-1)^\ell \frac{1}{\ell!} D_2 f(x,x) \dot{y}_{\ell-1} &= \tilde{F}(x, y_1, \ldots, y_{\ell-1}, \mu^{\ell-1}), \end{aligned} \tag{26}$$

where $\tilde{F}$ is obtained from the expansion of $\tau \mapsto f(x(t), x(t-\tau))$, which has the form

$$\begin{aligned} F(x, \dot{x}, \ldots, x^{(\ell-1)}, \tau) := {}& \dot{x} - f(x,x) + \tau D_2 f(x,x)\dot{x} \\ & - \frac{\tau^2}{2!}(D_2 f(x,x)\ddot{x} + D_2^2 f(x,x)(\dot{x},\dot{x})) + O(\tau^3), \end{aligned}$$

by rearranging terms and dividing out factors of the form $\mu^{(i-1)\ell}$ that appear in denominators when $y_i/\mu^{(i-1)\ell}$ is substituted for x^i in the arguments of the function F.

The family (26), parametrized by $\mu \in \mathbb{R}$, is singular at $\mu = 0$. It is transformed to a family that is regular at $\mu = 0$ by changing to the fast-time $s = t/\mu^\ell$. In fact, under the assumption that $D_2 f(x,x)$ is invertible, the fast-time family corresponding to system (26) is given by

$$\begin{aligned} x' &= \mu^{\ell} y_1, \\ y_1' &= y_2, \\ y_2' &= y_3, \\ &\vdots \\ y_{\ell-2}' &= y_{\ell-1}, \\ y_{\ell-1}' &= (-1)^{\ell} \ell! (D_2 f(x,x))^{-1} \tilde{F}(x, y_1, \ldots, y_{\ell-1}, \mu^{\ell-1}) \\ &= (-1)^{\ell} \ell! (D_2 f(x,x))^{-1} (y_1 - f(x,x) + O(\mu)). \end{aligned} \tag{27}$$

This fast-time system is equivalent to the slow-time system (26) only if $\mu \neq 0$. Thus, there are at least two reasons for the delicate nature of the smoothness at $\tau = 0$ of the slow flow, which will first be obtained using the fast-time system (27): the fast-time system is not equivalent to the original slow-time system at $\mu = 0$ and the relation between τ and μ (i.e. $\mu = \tau^{1/(\ell-1)}$) is not smooth at the origin.

Continuing with the usual procedure (see [9]), let us set $\mu = 0$ in system (27) and note that the resulting unperturbed system has an n-dimensional invariant manifold consisting entirely of rest points:

$$\mathcal{M}_0 := \{(x, y_1, y_2, \ldots, y_{\ell-1}) : y_1 = f(x,x), y_2 = 0, y_3 = 0, \ldots, y_{\ell-1} = 0\}.$$

This manifold is the graph of the C^{∞} function $S_0 : \mathbb{R}^n \to \mathbb{R}^{n(\ell-1)}$ given by $x \mapsto (f(x,x), 0, 0, \ldots, 0)$. Moreover, if ℓ is even and $D_2 f(x,x)$ is invertible, then the manifold ℓ, is normally hyperbolic, which (in our special case) means that the linearization at each rest point has exactly n eigenvalues on the imaginary axis (see [9]). This result is proved in [2, Prop. 3.1].

According to Fenichel (see Theorem 2 in [9]), if K is a compact subset of $\mathbb{R}^n$, then there is some interval $\mathcal{J}$ containing $\mu = 0$ and a function

$$S : K \times \mathcal{J} \to \mathbb{R}^{n(\ell-1)}$$

such that each manifold in the family

$$\mathcal{M}_{\mu} := \{(x, y_1, y_2, \ldots, y_{\ell-1}) : (y_1, y_2, \ldots, y_{\ell-1}) = S(x, \mu)\}$$

(parametrized by μ) is a normally hyperbolic (locally) invariant manifold for the corresponding system (27). Moreover, $S(x, 0) = S_0(x)$ and, for each positive integer r, the function S is class C^r.

The manifolds $\mathcal{M}_\mu$ are called slow-manifolds because for small $|\mu|$ the flow on $\mathcal{M}_\mu$ is a small perturbation of the flow on $\mathcal{M}_0$ that consists entirely of rest points. Here we are only interested in the normal hyperbolicity of $\mathcal{M}_0$ because this hypothesis ensures the existence of the smooth family of *invariant* (slow) manifolds $\mathcal{M}_\mu$. We will ignore the compactness assumption. In our application to the identity of inertial and slow flows, we can simply restrict attention to the identity of these functions at some arbitrary position $x_0 \in \mathbb{R}^n$ and time $t_0 \in \mathbb{R}$ and apply the Fenichel theory in a compact subset containing an open neighborhood of the orbit of the inertial flow starting at x_0 and continuing to time t_0.

The family of flows, corresponding to the family of ODEs (27), when restricted to the family of slow manifolds $\mathcal{M}_\mu$ is a family of slow flows in the sense of geometric singular perturbation theory. We will see that this family, viewed as a family parametrized by τ, is order-ℓ slow with respect to the family of delay equations (22).

The flow u of the fast-time system (27) is given in component form by

$$u(s,x,y,\mu) = (u_1(s,x,y,\mu), u_2(s,x,y,\mu), \dots, u_\ell(s,x,y,\mu)),$$

where $(x,y) \in \mathbb{R}^n \times \mathbb{R}^{(n-1)\ell}$ and s is the fast-time. Its restriction to $\mathcal{M}_\mu$, that is

$$(s,x,\mu) \mapsto u(s,x,S(x,\mu),\mu),$$

is the family of slow flows. In the local coordinates given by the x-space $\mathbb{R}^n$, the family of slow flows is given by

$$(s,x,\mu) \mapsto u_1(s,x,S(x,\mu),\mu);$$

it is the family of flows for the family of differential equations

$$x' = \mu^\ell S_1(x,\mu),$$

where $S_1 : K \times \mathcal{J} \to \mathbb{R}^n$ is the first component function associated with $S = (S_1, S_2, \dots, S_{\ell-1})$. In the original slow-time t and for $\mu \neq 0$, the local coordinate representation of the family of slow flows is given by

$$(t,x,\mu) \mapsto \psi(t,x,\mu) = u_1(t/\mu^\ell, x, S(x,\mu),\mu);$$

it is, for $\mu \neq 0$, the family of flows for the family of differential equations

$$\dot{x} = S_1(x,\mu).$$

Since the function S_1 is class C^r, jointly in x and μ for each $0 \leq r < \infty$, this last family of differential equations has a C^r family of flows that agree with ψ for each $\mu \neq 0$. In particular, ψ satisfies the ODE (23), which is equivalent to the singular first-order slow-time system (26). Let J denote the image of $\mathcal{J}$ under the map $\mu \to \mu^{\ell-1}$. It follows that $\psi(t, x, \tau^{1/(\ell-1)}) = \psi(t, x, \mu)$ satisfies the equation

$$\begin{aligned} \frac{\partial \psi}{\partial t}(t, \xi, \tau^{1/(\ell-1)}) = T_\ell(f)(\psi(t, \xi, \tau^{1/(\ell-1)}), \\ \tau^1 \frac{\partial \psi}{\partial t}(t, \xi, \tau^{1/(\ell-1)}), \ldots, \tau^\ell \frac{\partial^\ell \psi}{\partial t^\ell}(t, \xi, \tau^{1/(\ell-1)})) \end{aligned} \qquad (28)$$

for $\tau \in J$.

Proposition 5.2. *Suppose that ℓ is an even positive integer. The family of flows φ given by*

$$\varphi(t, x, \tau) := \psi(t, x, \tau^{1/(\ell-1)})$$

and, for each $i = 1, 2, \ldots, \ell$, the family of partial derivatives $\frac{\partial^i \varphi}{\partial t^i}$ are class $C^{\ell-1}$ on $K \times J$. Also,

$$\frac{\partial \varphi}{\partial t}(t, \xi, \tau) = T_\ell(f)(\varphi(t, \xi, \tau), \tau \frac{\partial \varphi}{\partial t}(t, \xi, \tau), \ldots, \tau^\ell \frac{\partial^\ell \varphi}{\partial t^\ell}(t, \xi, \tau)) \qquad (29)$$

for every $(t, \xi, \tau) \in \mathbb{R} \times K \times J$.

The first statement of Theorem 5.1 is an immediate corollary of Proposition 5.2 for $\ell = N$. Also, this proposition implies that the function

$$(t, x) \mapsto \frac{\partial^{j+1} \varphi}{\partial t \partial \tau^j}(t, \xi, \tau)$$

is continuous for $j = 0, 1, 2, \ldots, \ell - 1$. Thus, the partial derivatives mentioned in the last statement of Theorem 5.1 exist.

The second statement of Proposition 5.2 follows immediately from the definition of φ and equation (28). We will outline a proof of the required smoothness of φ on $K \times J$.

Let us consider the vector field $(x, \mu) \to S_1(x, \mu)$, the generator of the flow ψ.

Proposition 5.3. *The order-*$(\ell-1)^2$ *Taylor polynomial of the function* $\mu \to S_1(x,\mu)$ *has the form*

$$\sum_{i=0}^{\ell-1} \mu^{i(\ell-1)} S_{1,i(\ell-1)}(x).$$

A sketch of the proof of this result is given in [2, Claim 3.2].

By Proposition 5.3, the Taylor coefficients of $\mu \to S_1(x,\mu)$ vanish except those with order $i(\ell-1)$, where $i \in \{0,1,2,\ldots,(\ell-1)^2\}$. Because $\tau = \mu^{\ell-1}$, the Taylor polynomial is also a degree-$(\ell-1)$ polynomial in powers of τ.

By Proposition 5.3 and Taylor's theorem applied to $\mu \mapsto S_1(x,\mu)$,

$$\begin{aligned} S_1(x,\mu) &= \sum_{i=0}^{\ell-1} \mu^{i(\ell-1)} S_{1,i(\ell-1)}(x) + \mu^{(\ell-1)(\ell-1)} R(x,\mu) \\ &= \sum_{i=0}^{\ell-1} \tau^i S_{1,i(\ell-1)}(x) + \tau^{\ell-1} R(x,\tau^{1/(\ell-1)}), \end{aligned}$$

where $(x,\mu) \mapsto R(x,\mu)$ is as smooth as we like and $R(x,0)=0$ For our purposes, it suffices to have the function $(x,\mu) \mapsto R(x,\mu)$ be class $C^{2\ell-1}$.

The derivatives of the function $(x,\tau) \mapsto R(x,\tau^{1/(\ell-1)})$ are not defined at $\tau = 0$, but this singularity is mollified in the associated function $(x,\tau) \mapsto \tau^{\ell-1}R(x,\tau^{1/(\ell-1)})$. The next proposition gives a precise statement of this result for the function $\mathcal{S} : \mathbb{R}^n \times J \to \mathbb{R}^n$ given by

$$\mathcal{S}(x,\tau) := \sum_{i=0}^{\ell-1} \tau^i S_{1,i(\ell-1)}(x) + \tau^{\ell-1} R(x,\tau^{1/(\ell-1)}). \tag{30}$$

Because φ is the flow of $\dot{x} = \mathcal{S}(x,\tau)$, we will simultaneously determine the smoothness of the family of flows φ required to prove Proposition 5.2.

Proposition 5.4. *If* ℓ *is even,* r *and* p *are nonnegative integers, and* $R : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *is* C^{r+p} *on an open product neighborhood of the form* $U \times J$ *(where* J *could be an open interval containing* $\tau = 0$*), then the function given by*

$$(x,\tau) \mapsto \tau^r R(x,\tau^{1/(\ell-1)})$$

and its first p *partial derivatives with respect to* x *are* C^r *on* $U \times J$*. In particular, the function* $\mathcal{S}$ *and, for* $i = 1,2,\ldots,\ell$*, the partial derivative* $D_1^i\mathcal{S}$ *are both class* $C^{\ell-1}$ *on open subsets of* $\mathbb{R}^n \times \mathbb{R}$ *of the form* $U \times J$ *where* J *is an open interval containing* $\tau = 0$*.*

The proof of Proposition 5.4 is an exercise in calculus. The key point is that the power r of τ is sufficiently large to smooth the singularity at $\tau = 0$ that appears in the derivatives of the function $\tau \mapsto R(x, \tau^{1/(\ell-1)})$. Because $(x, \mu) \mapsto R(x, \mu)$ in equation (30) is class $C^{2\ell+1}$, the same exercise shows that, for $i = 1, 2, \ldots, \ell$, the function $(x, \tau) \mapsto \tau^{\ell-1} D_1^i R(x, \tau^{1/(\ell-1)})$ is class $C^{\ell-1}$.

To finish the proof of Proposition 5.2, recall that by definition we have

$$\frac{\partial \varphi}{\partial t}(t, x, \tau) = \mathcal{S}(\varphi(t, x, \tau), \tau). \tag{31}$$

Since, by Proposition 5.4, the function $\mathcal{S}$ is class $C^{\ell-1}$, its flow φ has the same smoothness. Also, it is immediate from equation (31) that the partial derivative of φ with respect to t is class $C^{\ell-1}$. By Proposition 5.4, the function $D_1\mathcal{S}$ is class $C^{\ell-1}$. Hence, the second partial derivative with respect to t enjoys the same smoothness because it satisfies the equation

$$\frac{\partial^2 \varphi}{\partial t^2}(t, x, \tau) = D_1\mathcal{S}(\varphi(t, x, \tau), \tau)\frac{\partial \varphi}{\partial t}(t, x, \tau)$$

whose right-hand side is class $C^{\ell-1}$. The $C^{\ell-1}$ smoothness of the higher order partial derivatives of φ with respect to t follows by induction.

Let us return to the proof of Theorem 5.1. We have already established the existence of a class C^{N-1} order-N slow flow u associated with the delay equation (22). Let $M \geq 2N+2$ be an even integer. By the same result, there is a class C^{M-1} order-M slow flow φ associated with the delay equation (22). We will show that the family of flows φ is $T_N(f)$-admissible.

Since $M - 1 \geq 2N + 1$, the family φ is class C^{2N+1}. We claim that it satisfies equation (23) with h a class C^N function. To prove this fact, note first that $M \geq 2N + 2 > N$; therefore, the Taylor polynomials $T_N(f)$ and $T_M(f)$ agree to order N. Thus, φ solves the ODE

$$\begin{aligned} x^{(1)} &= T_M(x, \tau^1 x^{(1)}, \tau^2 x^{(2)}, \ldots, \tau^M x^{(M)}) \\ &= T_N(x, \tau^1 x^{(1)}, \tau^2 x^{(2)}, \ldots, \tau^N x^{(N)}) + \tau^{N+1} Q(x, x^{(1)}, x^{(2)}, \ldots, x^{(M)}, \tau), \end{aligned}$$

where Q is class C^∞. Thus, it suffices to show that the function

$$(t, x, \tau) \mapsto Q(\varphi(t, \xi, \tau), \frac{\partial \varphi}{\partial t}(t, \xi, \tau), \ldots, \frac{\partial^M \varphi}{\partial t^M}(t, \xi, \tau), \tau)$$

is class C^N. In fact, this function is class C^{2N+1} because φ together with its partial derivatives with respect to t (up to order M) are all class C^{2N+1} and Q is class C^∞.

Finally, we will prove the last statement of Theorem 5.1.

The family of flows φ solves the ODE

$$x^{(1)} = T_N(x, \tau^1 x^{(1)}, \tau^2 x^{(2)}, \dots, \tau^N x^{(N)}) + \tau^{N+1} Q(x, x^{(1)}, x^{(2)}, \dots, x^{(M)}, \tau), \tag{32}$$

and the family u solves the ODE

$$x^{(1)} = T_N(x, \tau^1 x^{(1)}, \tau^2 x^{(2)}, \dots, \tau^N x^{(N)}). \tag{33}$$

The initial conditions for these flows are

$$\varphi(0, x, \tau) = x, \qquad u(0, x, \tau) = x,$$

and the initial conditions for their τ derivatives are

$$\frac{\partial^j \varphi}{\partial \tau^j}(0, x, \tau) = 0, \qquad \frac{\partial^j u}{\partial \tau^j}(0, x, \tau) = 0.$$

Proceeding inductively, when we differentiate the ODEs (32) and (33) j times with respect to τ, for $j = 1, 2, \dots, N-1$, and then set $\tau = 0$, we obtain identical *first-order* ODEs because the remainder term of ODE (32) is order $N+1$. The desired partial derivatives

$$\frac{\partial^j \varphi}{\partial \tau^j}(t, x, 0), \qquad \frac{\partial^j u}{\partial \tau^j}(t, x, 0),$$

are therefore solutions of the same first-order ODE with same initial condition. Hence, these partial derivatives are identical. This completes the outline of the proof of Theorem 5.1.

It is natural to ask: Why stop at order $N-1$? Answer: By the proof given here, we only know that u is class C^{N-1}.

References

[1] R. Abraham, J. Marsden, and T. Ratiu, *Manifolds, Tensor Analysis, and Applications,* 2nd Ed. New York: Springer-Verlag, 1988.

[2] C. Chicone, Inertial and slow manifolds for delay equations with small delays, *J. Diff. Eqs.*, **190** (2003), 364–406.

[3] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel and H. O. Walther, “Delay Equations: Functional-, Complex-, and Nonlinear Analysis,” Springer-Verlag, New York, 1995.

[4] N. Fenichel, Persistence and smoothness of invariant manifolds for flows, *Indiana Univ. Math. J.* **21** (1971), 193–226.

[5] N. Fenichel, Geometric singular perturbation theory for ordinary differential equation, *J. Diff. Eqs.*, **31** (1979), 53–98.

[6] H. Flanders, From Ford to Faà, *Amer. Math R. Mon.*, **108**(6) (2001), 559–561.

[7] M. Hirsch, C. Pugh, and M. Shub, “Invariant Manifolds”, Lecture Notes in Math., **583**. Springer-Verlag, New York, 1977.

[8] W. P. Johnson, The curious history of Faà di Bruno’s formula, *Amer. Math. Mon.*, **109**(3) (2002), 217–234.

[9] C. K. R. T. Jones, Geometric singular perturbation theory, *Lecture Notes in Math,* **1609**, Springer-Verlag, New York, 1995, 44–118.

[10] M. Petkovšek, H. S. Wilf, and D. Zeilberger, *A=B*, A K Peters, Ltd, Wellesley, MA, 1996.

[11] J. W. Robbin, On the existence theorem for differential equations, *Proc. Amer. Math. Soc.*, **19** (1968), 1005–1006.

[12] S. Parrott, Private communications, 2002–2003.