BIFURCATIONS OF NONLINEAR OSCILLATIONS AND FREQUENCY ENTRAINMENT NEAR RESONANCE *

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Abstract. We develop a unified approach to the Poincaré-Andronov global center bifurcation and the subharmonic Melnikov bifurcation theory using S. P. Diliberto's integration of the variational equations of a two dimensional system of autonomous ordinary differential equations and a Lyapunov-Schmidt reduction to the Implicit Function Theorem. In addition we generalize the subharmonic Melnikov function to the case of subharmonic bifurcation from an unperturbed system whose free oscillation is a limit cycle. Thus, we obtain results on frequency entrainment when an external periodic excitation is in resonance with the frequency of the limit cycle. The theory is applied to the subharmonic bifurcations of two coupled van der Pol oscillators running in resonance.

Key words. limit cycles, center bifurcations, subharmonics, Melnikov method, forced oscillations, frequency entrainment.

1. Introduction. The subject of this paper is the theory of frequency entrainment for driven nonlinear oscillators when the period of the self sustained free oscillation is nearly resonance with a periodic external excitation. Our main purpose is to demonstrate a mathematical theorem that is useful in determining the number and position of the subharmonics produced by such an external excitation. Our result is closely related to two well known theorems, the Poincaré-Andronov theorem on the global center bifurcation [1] and the Melnikov theorem on the global bifurcation of subharmonics from an integrable system [21, 42, 43]. In fact, our theorem can be considered as a generalization of Melnikov's method to cover the case of self sustained oscillations.

In order to explain the main result, consider a forced oscillation problem of the following type

$$\dot{x} = \mathbf{f}(x) + \epsilon\mathbf{g}(x,t), \quad x \in \mathbb{R}^2,\ \epsilon \in \mathbb{R}$$

where the unperturbed system

$$\dot{x} = \mathbf{f}(x)$$

with flow $t \mapsto \phi_t$ has a limit cycle Γ of period T as a self sustained oscillation, the external excitation is periodic of period η, i. e. ,

$$\mathbf{g}(x, t+\eta) = \mathbf{g}(x,t),$$

and the period of the external excitation is in resonance with the period of Γ, i. e. , there are relatively prime positive integers m and n such that $nT = m\eta$. We are interested in the periodic solutions of the forced system of period $m\eta$, the subharmonics of order m. For this, we look for a curve $\epsilon \mapsto \sigma(\epsilon)$ in the phase plane such

*This document was written April 24, 2003. This research was supported by the Air Force Office of Scientific Research and the National Science Foundation under the grants AF-AFOSR-89-0078 and DMS-9022621.

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that $\sigma(0) \in \Gamma$ and, for sufficiently small ϵ, such that the point $\sigma(\epsilon)$ is the initial value for a subharmonic of order m. When there is such a curve of initial conditions for a family of subharmonics, we say $\sigma(0)$ is a subharmonic branch point on Γ. Our main result gives a real valued function $\xi \mapsto \mathcal{C}(\xi)$, for ξ a coordinate on Γ, such that the simple zeros of $\mathcal{C}$ are the subharmonic branch points. The formula for this function is expressed in terms of Euclidean geometrical quantities that we denote as follows: $|| \; ||$ denotes the Euclidean norm, $<, >$ the Euclidean inner product, κ the scalar curvature, div the divergence of a vector field, curl the curl of a vector field, and $\wedge$ the wedge product of two vectors. In fact, the bifurcation function $\mathcal{C}$ is defined by

$$\mathcal{C}(\xi) := [(1-\beta)\mathcal{N} + \alpha\mathcal{M}]\,(m\eta, \xi)$$

where

$$\beta(t) := \beta(t,\xi) := \exp\left(\int_0^t \operatorname{div}\, \mathbf{f}(\phi_s(\xi))\, ds\right),$$

$$\alpha(t) := \alpha(t,\xi) := \int_0^t \left\{\frac{1}{||\mathbf{f}||^2}\left[2\kappa||\mathbf{f}|| - \operatorname{curl}\, \mathbf{f}\right]\right\}(\phi_\tau(\xi))\beta(\tau)\, d\tau,$$

$$\mathcal{N}(t,\xi) := \int_0^t \left\{\frac{1}{||\mathbf{f}||^2}\langle \mathbf{f}, \mathbf{g}\rangle - \frac{\alpha(s)}{\beta(s)}\mathbf{f}\wedge\mathbf{g}\right\}(\phi_s(\xi))\, ds,$$

$$\mathcal{M}(t,\xi) := \int_0^t \left\{\frac{1}{\beta(s)}\mathbf{f}\wedge\mathbf{g}\right\}(\phi_s(\xi))\, ds.$$

Our main theorem, the Limit Cycle Subharmonic Bifurcation Theorem, states: *If either $\beta(m\eta,\xi) \neq 1$ or $\alpha(m\eta,\xi) \neq 0$ and if ξ is a simple zero of $\mathcal{C}$, then ξ is a subharmonic branch point.*

If the periodic trajectory Γ of the unperturbed system is not a limit cycle but rather a periodic trajectory of period T contained in a one parameter family of periodic trajectories of the unperturbed system, then the appropriate bifurcation function is the subharmonic Melnikov function given by $\mathcal{M}$. In order to show that $\mathcal{C}$ reduces to $\mathcal{M}$ in this case, consider a Poincaré section Σ for the unperturbed system that is orthogonal to Γ at a point $\xi_0 \in \Sigma$ and define both the Poincaré return map and the transition time function on Σ. In §2 we will show the derivative of the return map, evaluated at the coordinate on Σ corresponding to ξ_0, is $\beta(T,\xi_0)$, while the derivative of the transition time function is $\alpha(T,\xi_0)$. Thus, for example, if Γ is a member of a one parameter family of periodic orbits, then Γ is not hyperbolic, equivalently, $\beta(nT,\xi) \equiv 1$, and $\mathcal{C}$ reduces to the usual subharmonic Melnikov function. Moreover, in this case, the appropriate nondegeneracy condition of the theorem, $\alpha(m\eta,\xi) \neq 0$, reduces to the condition that the period function for the one parameter family of periodic trajectories of the unperturbed system has a non zero derivative at Γ; this is the usual nondegeneracy condition for the subharmonic Melnikov theory, cf. [15, 21, 43].

As a typical application, consider two weakly coupled van der Pol oscillators of the form

$$\begin{aligned}
\dot{u} &= v \\
\dot{v} &= -u + \delta(1-u^2)v \\
\dot{x} &= \tau y \\
\dot{y} &= \tau(-x + \delta(1-x^2)y) + \epsilon u
\end{aligned}$$

where $\delta > 0$, ϵ is a small parameter and $\tau > 0$ is a rational number. We view the second oscillator as a driven oscillator where the periodic forcing function, $t \mapsto u(t)$, is

FIG. 1.1. *Computer generated graph of $\mathcal{C}$ vs ξ for weakly coupled van der Pol oscillators* $\dot{u} = v$, $\dot{v} = -u + 0.1(1 - u^2)v$, $\dot{x} = 0.5y$, $\dot{y} = 0.5(-x + 0.1(1 - x^2)y) + \epsilon u$ *with* $2:1$ *resonance.*

the output of the first oscillator. Since τ is rational, the frequency of the free oscillation of the second oscillator is in resonance with the frequency of the external excitation. We ask for the periodic response of the second oscillator when $\epsilon \neq 0$. For background material on forced oscillation problems we refer to [15, 27, 30, 22, 23, 25, 32, 34, 41] for classic treatments and to [21, 42, 43] for some of the latest results. As a typical calculation we fix $\delta = 0.1$ and $\tau = 0.5$. For this example the period of the free oscillation of the second oscillator is twice the period of the "external" force, a $2:1$ resonance. A numerical approximation to the graph of $\mathcal{C}(\xi)$ is shown in Figure 1. The graph indicates the existence of four simple zeros of $\mathcal{C}$ over one period of the free oscillation and, applying our theorem, we expect, for $|\epsilon|$, sufficiently small, four subharmonics of order two, i. e. , four periodic solutions of the forced second oscillator each with period twice the period of the self sustained oscillation of the first oscillator. Of course, in view of the topology of the circle, two of these subharmonics are stable and two of them are unstable, with the subharmonics corresponding to consecutive zeros of $\mathcal{C}$ having opposite stability.

Our treatment of the bifurcation theory of nonlinear oscillations is based on the geometric quadrature of the homogeneous variational equations of a two dimensional differential system given by S. P. Diliberto [20] and a Lyapunov-Schmidt reduction to the Implicit Function Theorem. Using this foundation, we are able to offer a unified approach to the bifurcation theory for plane vector fields that includes the Poincaré-Andronov center bifurcation theorem and the usual subharmonic Melnikov theorem. Since an exposition of these results requires only a minimum of additional effort, we do not offer the most efficient proof of our main result. Rather, we take a slightly longer route to our theorem that allows us to give new proofs of these other classic results. In addition to the unification provided by our Implicit Function Theorem approach to the bifurcation theory, we obtain smooth curves of bifurcating periodic solutions. Thus, the precise positions of the bifurcating solutions can be computed and, perhaps, continued to further global bifurcations. Also, we mention that our method yields proofs of the bifurcation theorems that do not require reduction to a normal form or a change to action angle variables.

The plan of the paper is as follows. In §2 we give a proof of Diliberto's Theorem and obtain the geometric interpretation of the functions α and β in terms of the Poincaré map and the transition time function. We also define the functions $\mathcal{N}$ and $\mathcal{M}$ in their natural context as constituents of the solution of a certain inhomogeneous variational equation. In §3 we prove the Poincaré-Andronov center bifurcation theorem. In §4 we prove the subharmonic Melnikov theorem, our main result, the Limit Cycle Subharmonic Bifurcation Theorem, and we prove some theorems on bifurcation of subharmonics from degenerate families. Also, we connect our theory with the classical perturbation theory for linear systems. In the final section, §5, we give additional applications of the theory.

2. Diliberto's Theorem. The fundamental result on which this paper is based is the theorem of Diliberto [20] on the integration of the homogeneous variational equations of a plane autonomous differential equation in terms of geometric quantities along a given trajectory of the system. Here we let $\mathbf{X} = (X_1, X_2)$ denote a smooth plane vector field with flow ϕ_t. The geometric quantities are the curvature, the curl and the divergence given by

$$\kappa = \|\mathbf{X}\|^{-3}\left(X_1\dot{X}_2 - X_2\dot{X}_1\right), \quad \operatorname{curl}\mathbf{X} = \frac{\partial X_2}{\partial x_1} - \frac{\partial X_1}{\partial x_2}, \quad \operatorname{div}\mathbf{X} = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2},$$

where $\|\mathbf{X}\| = \sqrt{\langle \mathbf{X}, \mathbf{X}\rangle}$ denotes the euclidean norm. It will also be convenient to define the orthogonal vector field $\mathbf{X}^{\perp} := (-X_2, X_1)$ as well as the vector field $\mathbf{U}_{\mathbf{X}^{\perp}}$ parallel to $\mathbf{X}^{\perp}$ given by

$$\mathbf{U}_{\mathbf{X}^{\perp}}(p) := \frac{1}{\|\mathbf{X}(p)\|^2}\mathbf{X}^{\perp}(p).$$

The normalization is chosen so that $\langle \mathbf{X}^{\perp}, \mathbf{U}_{\mathbf{X}^{\perp}}\rangle = 1$. Finally, we introduce the wedge product of two vector fields $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (X_1, X_2)$ to be $\mathbf{X} \wedge \mathbf{Y} = X_1Y_2 - X_2Y_1$. In the course of our discussion we make use of the formula

$$\mathbf{X} \wedge \mathbf{Y} = \langle \mathbf{X}^{\perp}, \mathbf{Y}\rangle .$$

It allows for a choice between two geometric interpretations of the same quantity, namely, the area given by the wedge product and the projection given by the inner product. This choice is rather arbitrary, but is often made according to tradition.

THEOREM 2.1 (Diliberto's Theorem). *If* $\mathbf{X}(p) \neq 0$, *then the linear variational equation along the integral curve* $t \mapsto \phi_t(p)$,

$$\dot{\mathbf{V}} = D\mathbf{X}(\phi_t(p))\mathbf{V},$$

has a fundamental matrix solution $\Phi(t)$, *satisfying* $\det(\Phi(0)) = 1$, *given by*

$$\Phi(t) := [\mathbf{X}(\phi_t(p)), \mathbf{V}(t)],$$

where

$$\mathbf{V}(t) := \alpha(t)\mathbf{X}(\phi_t(p)) + \beta(t)\mathbf{U}_{\mathbf{X}^{\perp}}(\phi_t(p))$$

and

$$\beta(t) := \beta(t, \mathbf{X}, p) := \exp\left(\int_0^t \operatorname{div}\, \mathbf{X}(\phi_s(p))\, ds\right),$$

$$\alpha(t) := \alpha(t, \mathbf{X}, p) := \int_0^t \left\{\frac{1}{\|\mathbf{X}\|^2}\left[2\kappa\|\mathbf{X}\| - \operatorname{curl}\, \mathbf{X}\right]\right\}(\phi_\tau(p))\beta(\tau)\, d\tau.$$

Moreover, the inverse of this fundamental matrix (partitioned by rows) is given by

$$\Phi^{-1}(t) = \frac{1}{\beta(t)}\begin{bmatrix} -\mathbf{V}^\perp(t) \\ \mathbf{X}^\perp(\phi_t(p)) \end{bmatrix}.$$

Proof. Define $\gamma(t) := \phi_t(p)$. Since $\mathbf{X}(p) \neq 0$, the function $t \mapsto \mathbf{X}(\gamma(t))$ is a nontrivial solution of the linear variational equation. Next, define

$$\mathbf{P}(t) := \frac{1}{\|\mathbf{X}(\gamma(t))\|}\left[\mathbf{X}(\gamma(t)), \mathbf{X}^\perp(\gamma(t))\right]$$

(partitioned by columns) and use the coordinate transformation $\mathbf{U} = \mathbf{P}^{-1}\mathbf{V}$ on the linear variational equation $\dot{\mathbf{V}} = D\mathbf{X}(\gamma(t))\mathbf{V}$ to obtain $\dot{\mathbf{U}} = \mathbf{A}\mathbf{U}$, where

$$\mathbf{A} := \mathbf{P}^{-1}(D\mathbf{X}(\gamma(t)))\mathbf{P} - \mathbf{P}^{-1}\dot{\mathbf{P}}.$$

A somewhat tedious calculation shows the matrix $\mathbf{A}$ is given by

$$\begin{aligned}\mathbf{A} &= \begin{bmatrix} \|\mathbf{X}\|^{-1}\frac{d}{dt}\|\mathbf{X}\| & 2\|\mathbf{X}\|^{-2}(X_1\dot{X}_2 - X_2\dot{X}_1) - \operatorname{curl}\mathbf{X} \\ 0 & -\|\mathbf{X}\|^{-1}\frac{d}{dt}\|\mathbf{X}\| + \operatorname{div}\mathbf{X} \end{bmatrix} \\ &= \begin{bmatrix} \frac{d}{dt}\ln\|\mathbf{X}\| & 2\|\mathbf{X}\|\kappa - \operatorname{curl}\mathbf{X} \\ 0 & -\frac{d}{dt}\ln\|\mathbf{X}\| + \operatorname{div}\mathbf{X} \end{bmatrix}.\end{aligned}$$

Since this system is in triangular form, we can find a simple representation for the general solution of $\dot{\mathbf{U}} = \mathbf{A}\mathbf{U}$. In fact, if we use $\mathbf{e}_1 := (1,0)$ and $\mathbf{e}_2 := (0,1)$, then

$$\mathbf{U}(t) = \frac{\|\mathbf{X}(\gamma(t))\|}{\|\mathbf{X}(p)\|}\left\{U_1(0) + U_2(0)\|\mathbf{X}(p)\|^2\alpha(t)\right\}\mathbf{e}_1 + \frac{U_2(0)\|\mathbf{X}(p)\|}{\|\mathbf{X}(\gamma(t)\|}\beta(t)\mathbf{e}_2,$$

where $\mathbf{U} = (U_1, U_2)$. Finally, if we choose the initial conditions so that one solution has initial condition $\mathbf{U}(0) = \|\mathbf{X}(p)\|\mathbf{e}_1$, and a second solution satisfies the initial condition $\|\mathbf{X}(p)\|\mathbf{U}(0) = \mathbf{e}_2$, then we obtain two linearly independent solutions. These solutions form the columns of a fundamental matrix $\widetilde{\Phi}$ for $\dot{\mathbf{U}} = \mathbf{A}\mathbf{U}$, while the fundamental matrix in the statement of the theorem is just $\Phi = \mathbf{P}\widetilde{\Phi}$. A simple calculation shows that $\det(\Phi(0)) = 1$. The statement about the inverse of the fundamental matrix $\Phi(t)$ is a straightforward deduction and its proof is omitted. □

REMARK: *It should be noted that the constant 2 multiplying the curvature in the theorem was omitted in the formulas in Diliberto's original paper.*

Diliberto's Theorem contains all the important information about the solutions of the linear variational equation along the trajectories of the plane vector field $\mathbf{X}$. We will use the theorem to derive several corollaries that illustrate the importance of the result. The first few of these corollaries can be taken to give the *geometric* meaning of the two functions α and β that are defined in the statement of the theorem. For this we introduce some basic definitions. Let p and q be points in $\mathbb{R}^2$ on the trajectory $t \mapsto \phi_t(p)$ of $\mathbf{X}$ with $q = \phi_\tau(p)$. We consider two sections for the flow of $\mathbf{X}$, Σ at p and Δ at q, given by plane curves $s \mapsto \sigma(s)$ and $s \mapsto \delta(s)$ with $\sigma(\xi) = p$. Moreover, we let $\mathbf{Y}$ denote the tangent vector field of σ and $\mathbf{Z}$ the tangent vector field of δ. We use $\widetilde{\mathbf{h}} : \Sigma \to \Delta$ to denote the *transition map*. It assigns to a point $r \in \Sigma$ the point where the trajectory of $\mathbf{X}$ starting from r first meets Δ. We use $\widetilde{T} : \Sigma \to \mathbb{R}$ to denote the *transition time function* that assigns to $r \in \Sigma$ the minimum positive time required for the transition. There will be an open interval $\mathcal{I} \subset \mathbb{R}$ such that for $s \in \mathcal{I}$ both $\widetilde{\mathbf{h}}$ and $\widetilde{T}$ are defined on $\sigma(\mathcal{I})$. We define $\mathcal{J} := \delta^{-1} \circ \widetilde{\mathbf{h}} \circ \sigma(\mathcal{I})$. Then, we can express the transition map and the transition time function in the local coordinates on the two sections defined by the functions σ and δ. It is convenient to give these representations names. In fact, we define the *scalar transition map* $h : \mathcal{I} \to \mathcal{J}$ by the formula

$$h := \delta^{-1} \circ \widetilde{\mathbf{h}} \circ \sigma,$$

and the *scalar transition time function* $T : \mathcal{I} \to \mathbb{R}$ by

$$T := \widetilde{T} \circ \sigma.$$

When we write $\widetilde{\mathbf{h}}'(p)$, we understand this to be the derivative, $h'(\xi)$ and when we write $\widetilde{T}'(p)$ we understand this to be the derivative $T'(\xi)$. Finally, we specify two special cases. If p is a periodic point of $\mathbf{X}$ we can take $\Sigma = \Delta$. In this case, the transition map is called the *return map* or the *Poincaré map* on the *Poincaré section* Σ. If, in addition, p is contained in a one parameter family of periodic trajectories of $\mathbf{X}$, we say p is in a *period annulus with (Poincaré) section* Σ. In this case, the corresponding transition time function is called the *period function*.

The next theorem identifies the functions α and β in Diliberto's Theorem geometrically and, in particular, provides formulas for the derivative of the return map and the period function.

THEOREM 2.2. *Let $\mathbf{X}$ be a plane vector field with flow ϕ_t, Σ a section for the flow at $p \in \mathbb{R}^2$ and Δ a section for the flow at $q = \phi_\tau(p)$. Also, let $s \mapsto \sigma(s)$ be a local coordinate function for Σ with tangent vector field $\mathbf{Y}$ such that $\sigma(\xi) = p$ and let $s \mapsto \delta(s)$ be a local coordinate function for Δ with tangent vector field $\mathbf{Z}$. If $\widetilde{\mathbf{h}}$ is the transition map and $\widetilde{T}$ is the transition time function, then*

$$\widetilde{\mathbf{h}}'(p) = \frac{\mathbf{X} \wedge \mathbf{Y}(p)}{\mathbf{X} \wedge \mathbf{Z}(\widetilde{\mathbf{h}}(p))} \beta(\widetilde{T}(p), \mathbf{X}, p)$$

and

$$\widetilde{T}'(p) = \frac{\langle \mathbf{Z}, \mathbf{X} \rangle}{||\mathbf{X}||^2}(\widetilde{\mathbf{h}}(p))\widetilde{\mathbf{h}}'(p) - \frac{\langle \mathbf{Y}, \mathbf{X} \rangle}{||\mathbf{X}||^2}(p) - \mathbf{X} \wedge \mathbf{Y}(p)\alpha(\widetilde{T}(p), \mathbf{X}, p).$$

In particular, if p is a periodic point of $\mathbf{X}$, the derivative of the return map is given by

$$\widetilde{\mathbf{h}}'(p) = \frac{\mathbf{X} \wedge \mathbf{Y}(p)}{\mathbf{X} \wedge \mathbf{Y}(\widetilde{\mathbf{h}}(p))} \beta(\widetilde{T}(p), \mathbf{X}, p)$$

and, in addition, if p is contained in a period annulus, the derivative of the period function is given by

$$\widetilde{T}'(p) = -\mathbf{X} \wedge \mathbf{Y}(p)\alpha(\widetilde{T}(p), \mathbf{X}, p).$$

Proof. By the definition of the transition map and the transition time function we have

$$\widetilde{\mathbf{h}}(\sigma(s)) = \phi_{\widetilde{T}(\sigma(s))}(\sigma(s)),$$

or, in local coordinates,

$$\delta(h(s)) = \phi_{T(s)}(\sigma(s)).$$

After differentiating both sides of the last equation and evaluating at $s = \xi$, we obtain the following formula for the derivative of the transition map:

$$h'(\xi)\mathbf{Z}(q) = \frac{d}{ds}\phi_{T(s)}(\sigma(s))\Big|_{s=\xi} = D\phi_{T(\xi)}(p)\mathbf{Y}(p) + T'(\xi)\mathbf{X}(q).$$

We will apply Diliberto's Theorem to obtain the required formulas. For this, observe that the function $t \mapsto \mathbf{V}(t)$ giving the second column of the Diliberto fundamental matrix of the linear variational equations along the trajectory $t \mapsto \phi_t(p)$ of $\mathbf{X}$ is the unique solution of the following linear variational initial value problem

$$\dot{\mathbf{V}} = D\mathbf{X}(\phi_t(p))\mathbf{V}, \quad \mathbf{V}(0) = \mathbf{U}_{\mathbf{X}^\perp}(p),$$

whose solution evaluated at $T(\xi)$ is

$$\mathbf{V}(T(\xi)) = \alpha(T(\xi), \mathbf{X}, p)\mathbf{X}(q) + \beta(T(\xi), \mathbf{X}, p)\mathbf{U}_{\mathbf{X}^\perp}(q).$$

Since $t \mapsto D\phi_t(p)\mathbf{U}_{\mathbf{X}^\perp}(p)$ is a solution of the same initial value problem, we have

$$D\phi_t(p)\mathbf{U}_{\mathbf{X}^\perp}(p) = \mathbf{V}(t)$$

and, of course, we also have

$$D\phi_t(p)\mathbf{X}(p) = \mathbf{X}(\phi_t(p)).$$

Moreover, there are real numbers a, b, c and d such that

$$\mathbf{Y}(p) = a\mathbf{X}(p) + b\mathbf{U}_{\mathbf{X}^\perp}(p), \quad \mathbf{Z}(q) = c\mathbf{X}(q) + d\mathbf{U}_{\mathbf{X}^\perp}(q).$$

In fact, these numbers can be expressed in terms of the given vectors as follows:

$$a = \frac{\langle \mathbf{Y}, \mathbf{X}\rangle}{||\mathbf{X}||^2}(p), b = \mathbf{X} \wedge \mathbf{Y}(p),$$
$$c = \frac{\langle \mathbf{Z}, \mathbf{X}\rangle}{||\mathbf{X}||^2}(q), d = \mathbf{X} \wedge \mathbf{Z}(q).$$

Now, we compute

$$D\phi_t(p)\mathbf{Y}(p) = a\mathbf{X}(\phi_t(p)) + b\mathbf{V}(t)$$

and, after the obvious substitutions into the formula for $h'(\xi)$, we obtain

$$(ch'(\xi) - T'(\xi) - a - b\alpha)\,\mathbf{X}(q) + (dh'(\xi) - b\beta)\,\mathbf{U}_{\mathbf{X}^\perp}(q) = 0.$$

The first part of the theorem follows immediately from the last equality and the linear independence of $\mathbf{X}(q)$ and $\mathbf{U}_{\mathbf{X}^\perp}(q)$.

From the definition of the return map we can assume $\Sigma = \Delta$, $q = \widetilde{\mathbf{h}}(p)$ and $\mathbf{Y} = \mathbf{Z}$. The derivative of the return map in the statement of the theorem is obtained by specializing the formula,

$$h'(\xi) = \frac{b}{d}\beta,$$

just derived for the derivative of the transition map.

For the derivative of the period function we have the same specialization and, in addition, the fact that p lies in a period annulus. The periodicity of p implies $p = q$ and, in turn, $a = c$, while the membership of p in a period annulus implies $\widetilde{\mathbf{h}}'(p) = 1$. Hence, in this case, $ch'(\xi) - a = 0$ and the formula obtained for the derivative of the transition time function reduces to the required formula for the period function,

$$T'(\xi) = -b\alpha.$$

□

For the majority of situations encountered in practice, when dealing with a spiral flow in the plane, a horizontal line segment can be chosen as a Poincaré section. In this case, the representation of the derivative of the return map and of the period function given in the last theorem takes a particularly simple form. If the horizontal line segment is an interval of the line with equation $y = k$, then, for example, the vector field $\mathbf{Y}$ may be taken to be the unit vector in the (positive) horizontal direction. If X_2 denotes the second component of the vector field $\mathbf{X}$ then, with $\gamma(t) := \phi_t(x,k)$, the representation of the derivative of the return map is

$$h'(x) = \frac{X_2(x,k)}{X_2(h(x),k)} \exp\left(\int_0^{T(x)} \operatorname{div} \mathbf{X}(\gamma(t))\,dt\right)$$

while the representation of the derivative of the period function is

$$T'(x) = X_2(x,k)\int_0^{T(x)} \left\{\frac{1}{\|\mathbf{X}\|^2}\left[2\kappa\|\mathbf{X}\| - \operatorname{curl}\mathbf{X}\right]\right\}(\gamma(\tau)) \exp\left(\int_0^{\tau} \operatorname{div}\mathbf{X}(\gamma(t))\,dt\right) d\tau.$$

These cases are often encountered in the applications.

The geometric identification of the function β is now clear. In fact, generally, if we consider a trajectory $t \mapsto \phi_t(p)$ of $\mathbf{X}$ and two sections Σ at p and Δ at $\phi_\tau(p)$, then $\beta(\tau, \mathbf{X}, p)$ is just the normalized derivative of the transition map. The normalization coefficient is the quotient of the two wedge products given in the theorem. The geometric identification of the function α is more complicated in the general case. But, if the two sections are orthogonal to the trajectory through p, then $\alpha(\tau, \mathbf{X}, p)$ is just the derivative of the transition time function normalized by the multiplicative factor $-\mathbf{X}\wedge\mathbf{Y}(p)$.

The next lemma gives an explicit formula for the solution of the inhomogeneous linear variational equations along a trajectory of $\mathbf{X}$; this formula will be needed in the perturbation theory to be developed later.

LEMMA 2.3 (Variation Lemma). *Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{b} = (b_1, b_2)$ denote smooth plane vector fields and let ϕ_t denote the flow of $\mathbf{X}$. If $p \in \mathbb{R}^2$ and $\mathbf{X}(p) \neq 0$, then the solution, $t \mapsto \mathbf{W}(t)$, of the initial value problem*

$$\dot{\mathbf{W}} = D\mathbf{X}(\phi_t(p))\mathbf{W} + \mathbf{b}(\phi_t(p)), \quad \mathbf{W}(0) = 0$$

is given by

$$\mathbf{W}(t) = [\mathcal{N}(t) + \alpha(t)\mathcal{M}(t)]\,\mathbf{X}(\phi_t(p)) + [\beta(t)\mathcal{M}(t)]\,\mathbf{U}_{\mathbf{X}^\perp}(\phi_t(p)),$$

where

$$\mathcal{N}(t) := \mathcal{N}(t, \mathbf{X}, \mathbf{b}, p) := \int_0^t \left\{ \frac{1}{||\mathbf{X}||^2} \langle \mathbf{X}, \mathbf{b} \rangle - \frac{\alpha(s)}{\beta(s)} \mathbf{X} \wedge \mathbf{b} \right\} (\phi_s(p))\, ds,$$
$$\mathcal{M}(t) := \mathcal{M}(t, \mathbf{X}, \mathbf{b}, p) := \int_0^t \left\{ \frac{1}{\beta(s)} \mathbf{X} \wedge \mathbf{b} \right\} (\phi_s(p))\, ds$$

and α, β are defined in the statement of Diliberto's Theorem. In addition, if m is a positive integer, p is a point in a period annulus of $\mathbf{X}$ with local section Σ given at p by the integral curve $s \mapsto \sigma(s)$ of $\mathbf{X}^\perp$ such that $\sigma(\xi) = p$, and if T denotes the scalar period function defined on Σ, then

$$\alpha(mT(\xi)) = -\frac{m}{||\mathbf{X}(p)||^2} T'(\xi), \quad \beta(mT(\xi)) = 1$$

and

$$W(mT(\xi)) = \left[\mathcal{N}(mT(\xi)) - \frac{m}{||\mathbf{X}(p)||^2} T'(\xi) \mathcal{M}(mT(\xi)) \right] \mathbf{X}(p) + \mathcal{M}(mT(\xi)) \mathbf{U}_{\mathbf{X}^\perp}(p).$$

Proof. Using variation of parameters, we have

$$\mathbf{W}(t) = \Phi(t) \int_0^t \Phi^{-1}(s) \mathbf{b}(s)\, ds,$$

where Φ is the Diliberto fundamental matrix. To evaluate this formula for $\mathbf{W}$ we first observe that

$$-\mathbf{V}^\perp(t) = \left\{ -\alpha(t) \mathbf{X}^\perp + \beta(t) \frac{1}{||\mathbf{X}||^2} \mathbf{X} \right\} (\phi_t(p)).$$

Then, using the formulas for the Diliberto fundamental matrix and its inverse given in Diliberto's theorem, we compute

$$\int_0^t \Phi^{-1}(s) \mathbf{b}(\phi_s(p))\, ds = \begin{bmatrix} \mathcal{N}(t) \\ \mathcal{M}(t) \end{bmatrix}.$$

The first statement of the lemma is now immediate.

For the second part of the lemma, we assume p is a periodic point of $\mathbf{X}$ and the coordinate for the section at p is given by the integral curve $s \mapsto \sigma(s)$ of $\mathbf{X}^\perp$ with $\sigma(\xi) = p$. Then, since p belongs to a period annulus, the return map is the identity on Σ and we conclude from the formula for the return map obtained above that

$$\beta(mT(\xi)) = 1.$$

Moreover, using this fact, together with a straightforward change of variables in the integral representation of $\alpha(mT(\xi))$, or, using the formula for $\widetilde{T}'$, we find

$$\alpha(mT(\xi)) = m\alpha(T(\xi)).$$

Thus, in view of the formula for the derivative of the period function given in the previous theorem,

$$\alpha(mT(\xi)) = -\frac{m}{||\mathbf{X}(p)||^2}T'(\xi).$$

After substitution of these identities into the formula for $\mathbf{W}(t)$ and a simple rearrangement of the terms, the final equality in the statement of the lemma is proved. $\square$

3. The Poincaré-Andronov Theorem. In order to state the main result of this section, we consider a plane vector field $(x,y) \mapsto \mathbf{X}(x,y,\epsilon)$ depending on the real small parameter ϵ. In case the phase portrait of the unperturbed system $\mathbf{X}_0(x,y) := \mathbf{X}(x,y,0)$ contains a period annulus $\mathcal{A}$, we seek to determine if there is a periodic trajectory Γ contained in $\mathcal{A}$ and a continuous family Γ_ϵ of periodic trajectories of $(x,y) \mapsto \mathbf{X}(x,y,\epsilon)$ such that $\Gamma_0 = \Gamma$.

For this bifurcation problem it is convenient to consider the differential equation corresponding to the vector field $(x,y) \mapsto \mathbf{X}(x,y,\epsilon)$ in the form

$$\dot{x} = P(x,y) + \epsilon p(x,y) + O(\epsilon^2), \quad \dot{y} = Q(x,y) + \epsilon q(x,y) + O(\epsilon^2).$$

Also, we let ϕ_t^ϵ denote the flow of $\mathbf{X}$. We can always arrange the coordinates so that a certain horizontal line segment $\Sigma : y = y_0$ is transverse to the flow of $\mathbf{X}_0$ in $\mathcal{A}$. Then, there is some $\epsilon_0 > 0$ such that $(x,y) \mapsto \mathbf{X}(x,y,\epsilon)$ is transverse to Σ for all ϵ satisfying $|\epsilon| < \epsilon_0$. We assume Γ is one of the periodic trajectories in $\mathcal{A}$ transverse to Σ and let ξ denote the usual distance coordinate along Σ. Then, both the scalar transition time function $(\xi,\epsilon) \mapsto T(\xi,\epsilon)$ and the scalar Poincaré return map $(\xi,\epsilon) \mapsto h(\xi,\epsilon)$ are defined on Σ. It is convenient in the analysis to define $\mathbf{X}_0^\perp$ to be the vector field with components $(-Q,P)$ and $\mathbf{H}(\xi,\epsilon)$ to be the vector $(d(\xi,\epsilon),0)$, where d is the *displacement function* defined by $d(\xi,\epsilon) := h(\xi,\epsilon) - \xi$. We also define the *normalized displacement function* F by

$$F(\xi,\epsilon) = \mathbf{X}_0^\perp(\xi,y_0) \cdot \mathbf{H}(\xi,\epsilon) = -Q(\xi,y_0)d(\xi,\epsilon).$$

There are two basic facts: $F(\xi,\epsilon) = 0$ if and only if the trajectory of $(x,y) \mapsto \mathbf{X}(x,y,\epsilon)$ through (ξ,y_0) is periodic and $F(\xi,0) \equiv 0$. If there is an $\epsilon_* > 0$ and a continuous function $\beta : (-\epsilon_*,\epsilon_*) \to \Sigma$ such that $F(\beta(\epsilon),\epsilon) \equiv 0$, then, for each ϵ in the domain of β, there is a periodic trajectory Γ_ϵ of the vector field $(x,y) \to \mathbf{X}(x,y,\epsilon)$ passing through the point $(\beta(\epsilon),y_0)$. In this case, we say *a continuous family of periodic trajectories of* $\mathbf{X}$ *emerges from the periodic trajectory* Γ_0.

The next result provides a means to identify the periodic trajectories in a period annulus of the unperturbed system from which a family of limit cycles emerges. It is a version of the theorem given in [1] and is the basic bifurcation theorem in this context. Our approach to the proof of this theorem is somewhat different from the development of the same result in [1]. For example, in [1, §32], the vector field family in which the bifurcation occurs is assumed to be analytic and, in [1, §33], a first integral must be constructed for the unperturbed conservative system. Here, we prove the result by analyzing an appropriate variational equation directly. The theorem has two main components: a reduction and an identification. Here, reduction refers to reducing the problem of the existence of a family of limit cycles to an application of the Implicit Function Theorem and identification refers to the identification of the appropriate partial derivatives in terms of the components of the vector field $\mathbf{X}$.

THEOREM 3.1. *Let $(x,y) \mapsto \mathbf{X}(x,y,\epsilon)$ denote a vector field with flow $t \mapsto \phi_t^\epsilon$ and corresponding differential equation*

$$\dot{x} = P(x,y) + \epsilon p(x,y) + O(\epsilon^2), \quad \dot{y} = Q(x,y) + \epsilon q(x,y) + O(\epsilon^2)$$

such that the corresponding unperturbed vector field $\mathbf{X}_0$ given by $(x,y) \to \mathbf{X}(x,y,0)$ has a period annulus with Poincaré section $\Sigma \subset \{(x,y) : y = y_0\}$.

(i) Reduction: If there is a point $\xi_0 \in \Sigma$ such that the corresponding normalized displacement function F satisfies $F_\epsilon(\xi_0,0) = 0$ and $F_{\xi\epsilon}(\xi_0,0) \neq 0$, then there is a periodic trajectory Γ of $\mathbf{X}_0$ meeting Σ at ξ_0 and a continuous family, Γ_ϵ, of periodic trajectories of $\mathbf{X}$ emerging from Γ. Moreover, for sufficiently small $\epsilon \neq 0$, the periodic trajectory Γ_ϵ is a limit cycle of the vector field $(x,y) \to \mathbf{X}(x,y,\epsilon)$. In fact, if $\epsilon F_{\xi\epsilon}(\xi_0,0)/Q(\xi_0,y_0) > 0$, then Γ_ϵ is asymptotically stable while if $\epsilon F_{\xi\epsilon}(\xi_0,0)/Q(\xi_0,y_0) < 0$, then Γ_ϵ is asymptotically unstable.

(ii) Identification: The partial derivative of the normalized displacement function with respect to the bifurcation parameter is given by

$$F_\epsilon(\xi,0) = \int_0^{T(\xi,0)} (Pq - Qp)\,(\gamma(t)) \exp\left(-\int_0^t \operatorname{div} \mathbf{X}_0(\gamma(s))\,ds\right) dt,$$

where $\gamma(t) := \phi_t^0(\xi,y_0)$ is the integral curve corresponding to the periodic trajectory Γ through (ξ,y_0). In addition, if $F_\epsilon(\xi_0,0) = 0$ for some $\xi_0 \in \Sigma$, then

$$\begin{aligned} F_{\xi\epsilon}(\xi_0,0) = &-Q(\xi_0,y_0)\Big\{ \operatorname{div} \mathbf{X}_0(\xi_0,y_0) T_\epsilon(\xi_0,0) \\ &+ \int_0^{T(\xi_0,0)} \operatorname{div}(p,q)(\gamma(t))\,dt \\ &+ \int_0^{T(\xi_0,0)} \frac{d}{d\epsilon} \operatorname{div} \mathbf{X}_0(\phi_t^\epsilon(\xi_0,y_0))\Big|_{\epsilon=0} dt \Big\}. \end{aligned}$$

(iii) If $\mathbf{X}_0$ is Hamiltonian ($\operatorname{div} \mathbf{X}_0 \equiv 0$), then for $\xi \in \Sigma$

$$F_\epsilon(\xi,0) = \int_0^{T(\xi,0)} (Pq - Qp)(\gamma(t))\,dt = -\int_\Omega \operatorname{div}(p,q)\,dxdy.$$

If, in addition, $F(\xi_0,0) = 0$, then

$$F_{\xi\epsilon}(\xi_0,0) = -Q(\xi_0,y_0) \int_0^{T(\xi_0,0)} \operatorname{div}(p,q)(\gamma(t))\,dt.$$

Proof. Since $F(\xi,0) \equiv 0$,

$$F(\xi,\epsilon) = \epsilon\,(F_\epsilon(\xi,0) + O(\epsilon)) := \epsilon G(\xi,\epsilon).$$

But then, from the hypotheses,

$$G(\xi_0,0) = F_\epsilon(\xi_0,0) = 0 \;\text{ and }\; G_\xi(\xi_0,0) = F_{\xi\epsilon}(\xi_0,0) \neq 0.$$

The Implicit Function Theorem applied to G implies the existence of the required function $\epsilon \mapsto \beta(\epsilon)$. For the stability of the perturbed limit cycles, just note that the displacement has the form

$$d(\xi,\epsilon) = \epsilon d_\epsilon(\xi,0) + O(\epsilon^2).$$

Thus, for sufficiently small ϵ, if $\epsilon d_{\epsilon\xi}(\xi_0, 0) < 0$, then $\xi \mapsto d(\xi, \epsilon)$ crosses the horizontal line segment Σ with negative slope as ξ increases through ξ_0. It is then immediate from the definition of the displacement that Γ_ϵ is a stable limit cycle. By the same argument the limit cycle will be unstable when $\epsilon d_{\epsilon\xi}(\xi_0, 0) < 0$. But,

$$\epsilon d_{\epsilon\xi}(\xi_0, 0) = -\epsilon \frac{F_{\epsilon\xi}(\xi_0, 0)}{Q(\xi_0, y_0)}$$

and therefore the statement of the theorem follows.

For the computation of the partial derivative $F_\epsilon(\xi, 0)$, we have

$$F_\epsilon(\xi, 0) = \mathbf{X}_0^\perp(\xi, y_0) \cdot \mathbf{H}_\epsilon(\xi, 0)$$

with $\mathbf{H}_\epsilon(\xi, 0) = (h_\epsilon(\xi, 0), 0)$. Thus, we must compute h_ϵ. For this, consider the integral curve $(x(t, \xi, \epsilon), y(t, \xi, \epsilon))$ of $(x, y) \mapsto \mathbf{X}(x, y, \epsilon)$ starting at the point (ξ, y_0) and let $T(\xi, \epsilon)$ denote the time of first return of this solution to Σ. Clearly, we have

$$x(T(\xi, \epsilon), \xi, \epsilon) = h(\xi, \epsilon), \quad y(T(\xi, \epsilon), \xi, \epsilon) = y_0,$$

and, after differentiation with respect to ϵ and an evaluation at $\epsilon = 0$, we obtain

$$\begin{aligned}
&\dot{x}(T(\xi, 0), \xi, 0)T_\epsilon(\xi, 0) + x_\epsilon(T(\xi, 0), \xi, 0) = h_\epsilon(\xi, 0) \\
&\dot{y}(T(\xi, 0), \xi, 0)T_\epsilon(\xi, 0) + y_\epsilon(T(\xi, 0), \xi, 0) = 0.
\end{aligned}$$

Define

$$\mathbf{W}(t) := (x_\epsilon(t, \xi, 0), y_\epsilon(t, \xi, 0)),$$

and then, using the abbreviations $T := T(\xi, 0)$, $T_\epsilon := T_\epsilon(\xi, 0)$, and $\mathbf{H}_\epsilon := \mathbf{H}_\epsilon(\xi, 0)$, we have

$$T_\epsilon \mathbf{X}_0(\gamma(T)) + \mathbf{W}(T) = \mathbf{H}_\epsilon = T_\epsilon \mathbf{X}_0(\xi, y_0) + \mathbf{W}(T).$$

Consequently

$$F_\epsilon(\xi, 0) = \mathbf{X}_0^\perp(\xi, y_0) \cdot \mathbf{W}(T).$$

In order to compute $\mathbf{W}$ we will solve an appropriate variational initial value problem. Since

$$x(0, \xi, \epsilon) = \xi, \;\; y(0, \xi, \epsilon) = y_0$$

we have,

$$x_\epsilon(0, \xi, \epsilon) = 0, \;\; y_\epsilon(0, \xi, \epsilon) = 0.$$

Thus, it is easy to see that $\mathbf{W}$ is the solution of the the following initial value problem

$$\dot{\mathbf{W}} = D\mathbf{X}_0(\gamma(t))\mathbf{W} + \mathbf{b}(t) \;, \;\mathbf{W}(0) = 0,$$

where $\mathbf{b}(t) = (p(\gamma(t)), q(\gamma(t)))$. An application of the second part of the Variation Lemma with $m = 1$ implies

$$F_\epsilon(\xi, 0) = \mathbf{X}_0^\perp(\xi, y_0) \cdot \mathbf{W}(T) = \mathcal{M}(T, \mathbf{X}, \mathbf{b}, \gamma(0)).$$

Thus

$$F_\epsilon(\xi,0)=\int_0^T (Pq-Qp)\,(\gamma(t))\exp\left(-\int_0^t \operatorname{div}\mathbf{X}_0(\gamma(\tau))\,d\tau\right)dt$$

as required.

To obtain the representation for $F_{\xi\epsilon}(\xi,0)$ under the hypothesis that $F_\epsilon(\xi_0,0)=0$, we do not compute the mixed partial derivative directly from the representation just obtained $F_\epsilon(\xi,0)$. Rather, we return to the definition of F and compute the partial derivatives from the formula

$$F(\xi,\epsilon)=\mathbf{X}_0^\perp(\xi,y_0)\cdot\mathbf{H}(\xi,\epsilon).$$

Since $\mathbf{X}_0^\perp$ does not depend on ϵ, we have

$$F_{\xi\epsilon}(\xi,0)=\mathbf{X}_0^\perp\cdot\mathbf{H}_{\xi\epsilon}+\mathbf{X}_{0\xi}^\perp\cdot\mathbf{H}_\epsilon(\xi,0).$$

But, by hypothesis, $\mathbf{H}_\epsilon(\xi_0,0)=0$. So the required derivative is given by

$$F_{\xi\epsilon}(\xi_0,0)=\mathbf{X}_0^\perp(\xi_0,y_0)\cdot\mathbf{H}_{\xi\epsilon}(\xi_0,0).$$

To compute the partial derivatives of $\mathbf{H}$ we use the previously given representation of the scalar return map. In the present case, this takes the form

$$h_\xi(\xi,\epsilon)=\frac{X_2(\xi,y_0,\epsilon)}{X_2(h(\xi,\epsilon),y_0,\epsilon)}\exp\left(\int_0^{T(\xi,\epsilon)}\operatorname{div}\mathbf{X}(\phi_t^\epsilon(\xi,y_0),\epsilon)\,dt\right).$$

Since

$$\mathbf{H}_{\xi\epsilon}(\xi_0,0)=(h_{\epsilon\xi}(\xi_0,0),0)$$

we need only calculate $h_{\epsilon\xi}$. First, using the hypotheses, $h(\xi,0)=\xi$ and $h_\epsilon(\xi,0)=0$, it is easy to verify that the derivative of the first factor of $h_\xi(\xi,\epsilon)$ with respect to ϵ vanishes at $\epsilon=0$ and the value of this factor at $(\xi_0,0)$ is unity. Thus, the required derivative is given by

$$\begin{aligned}h_{\epsilon\xi}(\xi_0,0)=\exp\left(\int_0^T\operatorname{div}\mathbf{X}_0(\gamma(t))\,dt\right)\Big\{&\operatorname{div}\mathbf{X}_0(\xi_0,y_0)T_\epsilon(\xi_0,0)\\&+\int_0^{T(\xi_0,0)}\frac{d}{d\epsilon}\operatorname{div}\mathbf{X}(\phi_t^\epsilon(\xi_0,y_0),\epsilon)\Big|_{\epsilon=0}dt\Big\}.\end{aligned}$$

Since the characteristic exponent of Γ vanishes, the exponential term is unity. We also have

$$\frac{d}{d\epsilon}\operatorname{div}\mathbf{X}(\phi_t^\epsilon(\xi_0,y_0),\epsilon)\Big|_{\epsilon=0}=\operatorname{div}(p,q)(\phi_t^0(\xi_0,y_0))+\frac{d}{d\epsilon}\operatorname{div}\mathbf{X}_0(\phi_t^\epsilon(\xi_0,y_0))\Big|_{\epsilon=0},$$

and it follows that

$$\begin{aligned}h_{\epsilon\xi}(\xi_0,0)=\operatorname{div}\mathbf{X}_0(\xi_0,y_0)T_\epsilon(\xi_0,0)&+\int_0^{T(\xi_0,0)}\operatorname{div}(p,q)(\gamma(t))\,dt\\&+\int_0^{T(\xi_0,0)}\frac{d}{d\epsilon}\operatorname{div}\mathbf{X}_0(\phi_t^\epsilon(\xi_0,y_0))\Big|_{\epsilon=0}dt\end{aligned}$$

as required.

The proof of (iii) is just an application of Green's Theorem. □

There is a large literature on the applications of the theorem. In most of these applications the most difficult problem is the computation of the integral for $F_\epsilon(\xi, 0)$ and the determination of its simple zeros. Some examples of such computations can be found in the references [1, 5, 10, 13, 19, 21, 27, 39, 43, 44]. A classic, but very simple application, can be made for the van der Pol oscillator with small damping. For this we have the system

$$\dot{x} = y, \;\; \dot{y} = -x + \epsilon(1 - x^2)y.$$

Here the unperturbed system is linear and the period annulus fills the entire punctured plane. The positive x-axis is a Poincaré section and, using the theorem, we compute

$$\begin{aligned} F_\epsilon(\xi, 0) &= \int_0^{2\pi} (1 - x^2)y^2 \, dt \\ &= \int_0^{2\pi} (1 - \xi^2 \cos^2 t)\xi^2 \sin^2 t \, dt \\ &= -\frac{\pi}{4}\xi^2(\xi^2 - 4). \end{aligned}$$

The bifurcation function has a unique simple zero at $\xi = 2$ and

$$\epsilon \frac{F_{\epsilon\xi}(2,0)}{Q(2,0)} = 2\epsilon\pi.$$

Thus, there is a continuous family of limit cycles emerging from the periodic trajectory $x(t) = 2\cos t \;\; y(t) = -2\sin t$ of the unperturbed system such that the family consists of stable limit cycles for $\epsilon > 0$ and unstable limit cycles for $\epsilon < 0$.

4. Subharmonic Bifurcation Theory. In this section we consider bifurcation to periodic solutions in the family E_ϵ of planar differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon \mathbf{G}(\mathbf{x}, t, \epsilon), \;\; \mathbf{x} \in \mathbb{R}^2, \;\; \epsilon \in \mathbb{R},$$

where both $\mathbf{f}$ and $\mathbf{G}$ are smooth functions, $\mathbf{G}$ is an η-periodic function, i.e.,

$$\mathbf{G}(\mathbf{x}, t + \eta, \epsilon) = \mathbf{G}(\mathbf{x}, t, \epsilon)$$

for all $\mathbf{x}$, t and ϵ, and where $\mathbf{G}$ has the form

$$\mathbf{G}(\mathbf{x}, t, \epsilon) = \mathbf{g}(\mathbf{x}, t) + \epsilon \mathbf{g}_R(\mathbf{x}, t, \epsilon)$$

with both $\mathbf{g}$ and $\mathbf{g}_R$ smooth functions of the indicated variables. We are interested in the bifurcation of periodic orbits from periodic trajectories Γ of the unperturbed system E_0 as $|\epsilon|$ increases from zero. For this we make one further assumption: The period of Γ is in $m : n$ *resonance* with the period of the forcing function $\mathbf{G}$, i.e., there are relatively prime positive integers m and n such that the period of Γ is $m\eta/n$. At the end of this section we show how to relax this assumption in the presence of a "detuning".

The idea of the bifurcation theory is to find the periodic trajectories as fixed points of the *parameterized Poincaré map* $\mathbf{P} : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ given by $(\xi, \epsilon) \mapsto \mathbf{x}(m\eta, \xi, \epsilon)$, where $t \mapsto \mathbf{x}(t, \xi, \epsilon)$ is the solution of E_ϵ satisfying the initial condition $\mathbf{x}(0, \xi, \epsilon) = \xi$.

In this interpretation, ξ is in the section $\Sigma = \mathbb{R}^2 \times 0$ for the flow considered on the manifold diffeomorphic to $\mathbb{R}^2 \times S^1$ obtained by identification of the time modulo $m\eta$. The basic property of the Poincaré map is that, for fixed ϵ, a solution of E_ϵ that starts at a fixed point of the function $\xi \mapsto \mathbf{P}(\xi, \epsilon)$ is a periodic solution of E_ϵ. For example, suppose $\mathbf{P}(\xi_0, \epsilon) = \xi_0$ and $\mathbf{x}(t, \xi_0, \epsilon)$ satisfies $\mathbf{x}(0, \xi_0, \epsilon) = \xi_0$. If we define $\mathbf{y}(t) := \mathbf{x}(t + m\eta, \xi_0, \epsilon)$, then

$$\begin{aligned}\dot{\mathbf{y}} &= \mathbf{f}(\mathbf{y}) + \epsilon \mathbf{G}(\mathbf{y}, t + m\eta, \epsilon) \\ &= \mathbf{f}(\mathbf{y}) + \epsilon \mathbf{G}(\mathbf{y}, t, \epsilon).\end{aligned}$$

Thus, $\mathbf{y}$ is a solution of E_ϵ with the initial condition $\mathbf{y}(0) = \mathbf{x}(m\eta, \xi_0, \epsilon) = \xi_0$ and, by uniqueness of the solutions of E_ϵ, we have $\mathbf{x}(t, \xi_0, \epsilon) = \mathbf{y}(t, \xi_0, \epsilon)$. It follows that $\mathbf{x}$ is a periodic function of its first variable. Also, if $\mathbf{x}$ is not a constant solution of E_ϵ, the minimum period of $\mathbf{x}$ must be $m\eta/k$ for some positive integer k.

The Poincaré map is easy to compute on the resonant orbit Γ. Since, for the flow ϕ_t of the unperturbed system, we have

$$\phi_{m\eta}(\xi) = \phi_{n(m\eta/n)}(\xi) = \xi,$$

the function $\xi \mapsto \mathbf{P}(\xi, 0)$ is the identity on Γ. We wish to find conditions on the functions $\mathbf{f}$ and $\mathbf{G}$ such that, for sufficiently small $\epsilon \neq 0$, some of these fixed points remain. In order to state the bifurcation theorems that provide these conditions, we will need a few more definitions. We identify $\Sigma = \mathbb{R}^2 \times 0$ with $\mathbb{R}^2$ and, for $\xi \in \Sigma$, we define the *displacement function* $\delta(\xi, \epsilon) := \mathbf{P}(\xi, \epsilon) - \xi$ together with its *radial projection* $\rho(\xi, \epsilon) := \left\langle \delta(\xi, \epsilon), \mathbf{f}^\perp(\xi) \right\rangle$ and its *tangential projection* $\tau(\xi, \epsilon) := \langle \delta(\xi, \epsilon), \mathbf{f}(\xi) \rangle$, where $<,>$ denotes the usual inner product on $\mathbb{R}^2$. For Γ a periodic trajectory of the unperturbed system whose period is in resonance with the external periodic force $\mathbf{G}$, we say $\xi \in \Gamma$ is a *subharmonic branch point* if there is an $\epsilon_0 > 0$ and a curve, $\epsilon \mapsto \sigma(\epsilon)$, defined for $|\epsilon| < \epsilon_0$, with image in the section Σ, such that $\sigma(0) = \xi$ and $\delta(\sigma(\epsilon), \epsilon) \equiv 0$. Of course, if $\delta(\sigma(\epsilon), \epsilon) = 0$, then $\sigma(\epsilon) \in \Sigma$ is the initial value for a periodic solution of E_ϵ. When the unperturbed periodic solution Γ is in $m : 1$ resonance with the forcing, the resonance is called *subharmonic of order* m and the perturbed periodic solutions are called subharmonics. Subharmonic resonance of order one is also called *harmonic*. This is the reason for the use of the term "subharmonic" in the definition of subharmonic branch points. However, the bifurcating solutions at a subharmonic branch point may not be, in the strict sense of the term, subharmonics. For example, if $n \neq 1$, a $1 : n$ resonance is called *ultraharmonic* and an $m : n$ resonance is called *ultrasubharmonic*. See [43][pp. 73–78] for a geometric view of these solutions.

As in the Poincaré-Andronov Theorem, the bifurcation analysis for subharmonic branch points consists of two main steps: reduction and identification. Here, reduction refers to reducing the problem of the existence of a curve of subharmonics bifurcating from Γ to an application of the Implicit Function Theorem, i.e., to the nonvanishing of a certain partial derivative, while identification refers to finding an explicit formula for this derivative in terms of the functions $\mathbf{f}$ and $\mathbf{G}$. For this we will use throughout the functions defined in §2 and given by

$$\begin{aligned}
\beta(t) := \beta(t,\xi) &:= \exp\left(\int_0^t \operatorname{div}\, \mathbf{f}(\phi_s(\xi))\, ds\right),\\
\alpha(t) := \alpha(t,\xi) &:= \int_0^t \left\{\frac{1}{\|\mathbf{f}\|^2}\left[2\kappa\|\mathbf{f}\| - \operatorname{curl}\, \mathbf{f}\right]\right\}(\phi_\tau(\xi))\beta(\tau)\, d\tau,\\
\mathcal{N}(\xi) &:= \int_0^{m\eta} \left\{\frac{1}{\|\mathbf{f}\|^2}\langle \mathbf{f}, \mathbf{g}\rangle - \frac{\alpha(s)}{\beta(s)}\mathbf{f}\wedge\mathbf{g}\right\}(\phi_s(\xi))\, ds,\\
\mathcal{M}(\xi) &:= \int_0^{m\eta} \left\{\frac{1}{\beta(s)}\mathbf{f}\wedge\mathbf{g}\right\}(\phi_s(\xi))\, ds.
\end{aligned}$$

Also, we will compute several times the derivatives of functions of the two variables ξ and ϵ. We will use the convention that derivatives indicated by D, for functions with range in $\mathbb{R}^2$, or by d, for functions with range in $\mathbb{R}$, refer to the derivative with respect to the space variable ξ while derivatives indicated by a subscripted variable refer to the partial derivative with respect to that variable.

To begin the analysis, we expand the displacement function into its perturbation series

$$\begin{aligned}
\delta(\xi,\epsilon) &= \mathbf{P}(\xi,0) - \xi + \mathbf{P}_\epsilon(\xi,0)\epsilon + O(\epsilon^2)\\
&= \mathbf{x}(m\eta,\xi,0) - \xi + \mathbf{x}_\epsilon(m\eta,\xi,0)\epsilon + O(\epsilon^2)
\end{aligned}$$

and we recall that, by the Variation Lemma, the first order term of the perturbation series is given by

$$\mathbf{x}_\epsilon = (\mathcal{N} + \alpha\mathcal{M})\mathbf{f} + \beta\mathcal{M}\mathbf{u}_{\mathbf{f}^\perp}.$$

For ξ on the resonant orbit Γ, $\delta(\xi,0)\equiv 0$ and, consequently,

$$D\delta(\xi,0)(\mathbf{f}(\xi)) = \frac{d}{dt}\mathbf{P}(\phi_t(\xi),0)\bigg|_{t=0} - \mathbf{f}(\xi) = 0.$$

Thus, $\xi\mapsto D\delta(\xi,0)$ is not invertible and we can not use the Implicit Function Theorem directly. However, we can use various forms of the Lyapunov-Schmidt reduction depending on how degenerate the curve Γ is as part of the zero set of the function $\xi\to\delta(\xi,0)$.

The most degenerate case is the classical case when the unperturbed system is linear and $\delta(\xi,0)\equiv 0$. Actually, this degeneracy is equivalent to assuming Γ is contained in an isochronous period annulus, cf. [13]. In this case, we have the following proposition.

PROPOSITION 4.1. *Let* $\mathbf{P}$ *denote the parameterized Poincaré map for the system* E_ϵ,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon\mathbf{G}(\mathbf{x},t,\epsilon),\ \mathbf{x}\in\mathbb{R}^2,\ \epsilon\in\mathbb{R},$$

where $\mathbf{G}$ *is* η*-periodic of the form*

$$\mathbf{G}(\mathbf{x},t,\epsilon) = \mathbf{g}(\mathbf{x},t) + \epsilon\mathbf{g}_R(x,t,\epsilon)$$

and assume Γ *is a periodic solution of the unperturbed system which belongs to an isochronous period annulus.*

Reduction: If there are positive integers m *and* n *such that the period of* Γ *is equal to* $m\eta/n$ *and if the function* $\xi \mapsto \mathbf{P}_\epsilon(\xi, 0)$ *has a simple zero at* ξ_0*, i. e. ,*

$$\mathbf{P}_\epsilon(\xi_0, 0) = 0 \quad \textit{and} \quad det(D\mathbf{P}_\epsilon(\xi_0, 0)) \neq 0,$$

then ξ_0 *is a subharmonic branch point.*

Identification: The bifurcation function $\xi \mapsto \mathbf{P}_\epsilon(\xi, 0)$ *is given by*

$$\mathbf{P}_\epsilon(\xi, 0) = \left[\mathcal{N}\mathbf{f} + \frac{1}{||\mathbf{f}||^2}\mathcal{M}\mathbf{f}^\perp\right](\xi).$$

Moreover, in case the unperturbed system is linear with $\mathbf{f}(x_1, x_2) = (\omega x_2, -\omega x_1)$ *and if* $2\pi n/\omega = m\eta$*, then the bifurcation function is given by*

$$\mathbf{P}_\epsilon(\xi, 0) = \frac{1}{||\xi||^2}\left(\xi_1 I_2(\xi) + \xi_2 I_1(\xi),\, \xi_2 I_2(\xi) - \xi_1 I_1(\xi)\right)$$

where

$$I_1(\xi) := \int_0^{2\pi n/\omega} x_2 g_1(\mathbf{x}, t) - x_1 g_2(\mathbf{x}, t)\, dt, \quad I_2(\xi) := \int_0^{2\pi n/\omega} x_1 g_1(\mathbf{x}, t) + x_2 g_2(\mathbf{x}, t)\, dt.$$

Proof. By the hypotheses, the displacement function, $\delta(\xi, \epsilon) := \mathbf{P}(\xi, \epsilon) - \xi$, for the perturbed system can be represented in the form

$$\delta(\xi, \epsilon) = \epsilon\left[\mathbf{P}_\epsilon(\xi, 0) + O(\epsilon)\right].$$

Therefore, the Implicit Function Theorem can be applied to determine when there is an implicit solution of the equation $\mathbf{P}_\epsilon(\xi, 0) + O(\epsilon) = 0$ at some point $(\xi_0, 0)$. This proves the reduction statement of the proposition. For the identification we simply observe that Γ is not hyperbolic and that the period function on the period annulus is constant. Then, using the results of §2, we have $\beta(m\eta, \xi) \equiv 1$ and $\alpha(m\eta, \xi) \equiv 0$ for $\xi \in \Gamma$. Thus, from the formula for $\mathbf{x}_\epsilon$ given above, we obtain the desired result.

In case the unperturbed vector field $\mathbf{f}$ is linear, with $\mathbf{f}(x_1, x_2) = (\omega x_2, -\omega x_1)$, the punctured phase plane of the unperturbed system is filled by periodic trajectories each of which lies on a circle centered at the origin and the hypotheses of the proposition hold. If, in addition, we assume the period of the external excitation is in resonance with the linear system, i. e. , there are positive integers m and n such that $2\pi n/\omega = m\eta$, then the formula for $\mathbf{x}_\epsilon(m\eta, \xi, 0)$ reduces to

$$\frac{1}{\omega^2||\xi||^2}\left[\left(\int_0^{m\eta}\langle\mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x}, t)\rangle\, dt\right)\mathbf{f}(\xi) + \left(\omega\int_0^{m\eta}\mathbf{f}(\mathbf{x}) \wedge \mathbf{g}(\mathbf{x}, t)\, dt\right)\xi\right]$$

where, in components, $\xi = (\xi_1, \xi_2)$ and

$$\mathbf{x}(t) := (x_1(t), x_2(t)) = (\xi_1\cos(\omega t) + \xi_2\sin(\omega t),\, -\xi_1\sin(\omega t) + \xi_2\cos(\omega t)).$$

Moreover, if the components of the external excitation are given by

$$\mathbf{g}(\mathbf{x}, t) := (g_1(\mathbf{x}, t), g_2(\mathbf{x}, t)),$$

and we define

$$I_1(\xi) := \int_0^{2\pi n/\omega} x_2 g_1(\mathbf{x}, t) - x_1 g_2(\mathbf{x}, t)\, dt, \quad I_2(\xi) := \int_0^{2\pi n/\omega} x_1 g_1(\mathbf{x}, t) + x_2 g_2(\mathbf{x}, t)\, dt.$$

Then, we have

$$\mathbf{x}_\epsilon(m\eta, \xi, 0) = \frac{1}{||\xi||^2}\left(\xi_1 I_2(\xi) + \xi_2 I_1(\xi),\, \xi_2 I_2(\xi) - \xi_1 I_1(\xi)\right)$$

as required. □

For computational purposes, it is convenient to define $\mathcal{I}(\xi) := (I_1(\xi), I_2(\xi))$, and to observe

$$\mathbf{P}_\epsilon(\xi, 0) = \begin{pmatrix} \xi_2 & \xi_1 \\ -\xi_1 & \xi_2 \end{pmatrix} \begin{pmatrix} I_1(\xi) \\ I_2(\xi) \end{pmatrix}.$$

Then, the existence of a simple zero of $\mathbf{P}_\epsilon(\xi, 0)$ is easily seen to be equivalent to the existence of a simple zero of $\mathcal{I}$. A final special case, where we take $\omega = 1$ and $g_1(\mathbf{x}, t) \equiv 0$, results in the identification

$$\mathbf{P}_\epsilon(\xi, 0) = \left(-\int_0^{2\pi n} g_2(\mathbf{x}(t), t) \sin t\, dt,\, \int_0^{2\pi n} g_2(\mathbf{x}(t), t) \cos t\, dt\right).$$

Here, the components of $\mathbf{P}_\epsilon(\xi, 0)$ are given by the same formulas obtained from the classical perturbation series methods. See [27, XII, §2] for a version of the classic approach to these formulas and for the computations, using the same formulas, showing the existence of a unique stable harmonic solution of the forced van der Pol oscillator,

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + \epsilon(1 - x^2)y + \epsilon a \sin t. \end{aligned}$$

When $\xi \to \delta(\xi, 0)$ is not identically zero on some neighborhood of Γ, the bifurcation theory must deal with the zero order terms of the perturbation expansion of the displacement function. The least degenerate case occurs when the kernel $\mathcal{K}$ of the map $D\delta(\xi, 0) : \mathbb{R}^2 \to \mathbb{R}^2$ is one dimensional. We have already shown $\mathbf{f}(\xi) \in \mathcal{K}$. Thus, this nondegeneracy condition will be satisfied when $\mathbf{f}^\perp(\xi) \notin \mathcal{K}$. It turns out that this nondegeneracy condition is equivalent to having one of the following: either Γ is hyperbolic or, at $\xi \in \Gamma$, the transit time map on a section in the phase plane orthogonal to Γ has a nonzero derivative along the section at ξ. The second possibility is the only way to have nondegeneracy when Γ is in a period annulus. In this case, the transit time reduces to the period function. Of course, if the period function has a zero derivative "at Γ" on a section transverse to Γ, then this derivative will be zero on every such section. When the derivative of the period function vanishes in this way, Γ is called *critical*. However, it is worth noting that the vanishing of the derivative of the transit time along a section intersecting a limit cycle, depends on the choice of section.

For the next theorem we assume the unperturbed system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has flow ϕ_t and a noncritical periodic trajectory Γ given by $t \mapsto \phi_t(p)$ that lies in a period annulus $\mathcal{A} \subset \Sigma$. We let $T : \mathcal{A} \to \mathbb{R}$ denote the *period function* for the unperturbed system on the period annulus $\mathcal{A}$; it assigns to each $\xi \in \mathcal{A} \subset \Sigma$ the minimum period of the periodic trajectory of the unperturbed system passing through ξ. Also, the *subharmonic Melnikov function* is defined for $\xi \in \mathcal{A}$, in terms of the radial projection of the displacement function ρ, by

$$M(\xi) := \rho_\epsilon(\xi, 0).$$

THEOREM 4.2 (Subharmonic Bifurcation Theorem). *Let E_ϵ denote the parameterized family of differential equations*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon \mathbf{g}(\mathbf{x}, t) + \epsilon^2 \mathbf{g}_R(\mathbf{x}, t, \epsilon), \ \mathbf{x} \in \mathbb{R}^2, \ \epsilon \in \mathbb{R},$$

such that E_0 has flow $t \mapsto \phi_t$, a period annulus $\mathcal{A}$ and a periodic trajectory $\Gamma \subset \mathcal{A}$ that is in resonance with the η-periodic external force $\mathbf{G}(x, t, \epsilon) := \mathbf{g}(\mathbf{x}, t) + \epsilon \mathbf{g}_R(\mathbf{x}, t, \epsilon)$, i. e. , there are relatively prime natural numbers m and n such that the period of Γ is $m\eta/n$.

(i) Reduction: If Γ is not critical and $\xi \in \Gamma$ is a simple zero of the subharmonic Melnikov function, i.e., $dT(\mathbf{f}^\perp)(\xi) \neq 0$, $M(\xi) = 0$ and $dM(\mathbf{f})(\xi) \neq 0$, then ξ is a subharmonic branch point.

(ii) Identification: The directional derivative of the period function T in the direction $\mathbf{f}^\perp(\xi)$ is given by

$$dT(\mathbf{f}^\perp)(\xi) = -||\mathbf{f}(\xi)||^2 \int_0^{T(\xi)} \left\{ \frac{1}{||\mathbf{f}||^2} [2\kappa ||\mathbf{f}|| - \operatorname{curl} \mathbf{f}] \right\} (\phi_s(\xi)) \exp \left(\int_0^s \operatorname{div} \mathbf{f}(\phi_t(\xi))\, dt \right) ds$$

and the subharmonic Melnikov function is given by

$$M(\xi) = \mathcal{M}(m\eta, \mathbf{f}, \mathbf{g}, \xi) = \int_0^{m\eta} \exp \left\{ -\int_0^t \operatorname{div}\ \mathbf{f}(\phi_s(\xi))\, ds \right\} \mathbf{f}(\phi_t(\xi)) \wedge \mathbf{g}(\phi_t(\xi), t)\, dt.$$

Proof. We have

$$\delta(\xi, \epsilon) = \mathbf{P}(\xi, 0) - \xi + \mathbf{P}_\epsilon(\xi, 0)\epsilon + O(\epsilon^2).$$

For ξ on the resonant orbit Γ, $\delta(\xi, 0) \equiv 0$ and, as we have seen,

$$D\delta(\xi, 0)(\mathbf{f}(\xi)) = \frac{d}{dt} \mathbf{P}(\phi_t(\xi), 0) \Big|_{t=0} - \mathbf{f}(\xi) = 0.$$

Thus, $\xi \mapsto D\delta(\xi, 0)$ is not invertible and we can not use the Implicit Function Theorem directly. However, the Lyapunov–Schmidt reduction can be employed. First, we apply the Implicit Function Theorem to the tangential projection τ. For this, let $\xi \in \Gamma$ and compute the directional derivative of $\xi \mapsto \tau(\xi, 0)$ in the direction $\mathbf{f}^\perp$ to obtain

$$d\tau(\xi, 0)(\mathbf{f}^\perp(\xi)) = \left\langle D\delta(\xi, 0)\mathbf{f}^\perp(\xi), \mathbf{f}(\xi) \right\rangle + \left\langle \delta(\xi, 0), D\mathbf{f}(\xi)\mathbf{f}^\perp(\xi) \right\rangle.$$

Since $\delta(\xi, 0) \equiv 0$ on Γ, the formula for this derivative reduces to

$$d\tau(\xi, 0)(\mathbf{f}^\perp(\xi)) = \left\langle D\delta(\xi, 0)\mathbf{f}^\perp(\xi), \mathbf{f}(\xi) \right\rangle.$$

In order to compute $D\delta(\xi, 0)\mathbf{f}^\perp(\xi)$, recall

$$DP(\xi, 0)\mathbf{f}^\perp(\xi) = D\phi_{m\eta}(\xi)\mathbf{f}^\perp(\xi)$$

and let $\mathbf{V}$ be as defined in Diliberto's Theorem. Since both

$$t \mapsto D\phi_t(\xi) \frac{1}{||\mathbf{f}(\xi)||^2} \mathbf{f}^\perp(\xi) \ \text{ and } \ t \mapsto \mathbf{V}(t, \mathbf{f}, \xi)$$

are solutions of the homogeneous variational equation for the unperturbed system $\dot{x} = \mathbf{f}(x)$ along the trajectory $t \mapsto \phi_t(\xi)$ satisfying the same initial condition, the two functions are equal and we have

$$D\phi_{m\eta}(\xi)\mathbf{f}^{\perp}(\xi) = ||\mathbf{f}(\xi)||^2\mathbf{V}(m\eta) = ||\mathbf{f}(\xi)||^2 \left\{ \alpha(m\eta)\mathbf{f}(\xi) + \frac{\beta(m\eta)}{||\mathbf{f}(\xi)||^2}\mathbf{f}^{\perp}(\xi) \right\}.$$

In view of the representation of the period function given in the Variation Lemma and the nonhyperbolicity of Γ, this formula can be expressed more concisely as

$$D\phi_{m\eta}(\xi)\mathbf{f}^{\perp}(\xi) = -\left[m\, dT(\mathbf{f}^{\perp})(\xi)\right]\mathbf{f}(\xi) + \mathbf{f}^{\perp}(\xi)$$

and, in turn, we have a simple expression for the directional derivative of δ :

$$D\delta(\xi, 0)\mathbf{f}^{\perp}(\xi) = (D\mathbf{P}(\xi, 0) - I)\mathbf{f}^{\perp}(\xi) = -\left[m\, dT(\xi)(\mathbf{f}^{\perp}(\xi))\right]\mathbf{f}(\xi).$$

Taking the inner product with $\mathbf{f}(\xi)$, we find the required directional derivative of τ to be

$$d\tau(\xi, 0)(\mathbf{f}^{\perp}(\xi)) = -m\, ||\mathbf{f}(\xi)||^2\, dT(\xi)(\mathbf{f}^{\perp}(\xi)) \neq 0.$$

Now, by an application of the Implicit Function Theorem to the function τ, we conclude there is a smooth two dimensional surface $\mathcal{S}$ in the (ξ, ϵ)-space passing through the curve $\Gamma \times 0$ such that τ vanishes on $\mathcal{S}$. Since, in addition, for $\xi \in \Gamma$ we have

$$d\tau(\xi, 0)\mathbf{f}(\xi) = \langle (D\mathbf{P}(\xi, 0) - I)\mathbf{f}(\xi), \mathbf{f}(\xi)\rangle \equiv 0,$$

$\mathcal{S}$ is transverse to the section Σ and $\Gamma \subset \mathcal{S}$.

To complete the reduction we restrict our attention to the manifold $\mathcal{S}$. To be more precise, we consider a neighborhood of the point $(\xi_0, 0)$ on Γ. There is a local coordinate chart (U, φ_U) on $\mathcal{S}$ such that U is a product neighborhood in $\mathbb{R} \times \mathbb{R}$ and $\varphi_U : U \to \mathcal{S} \subset \mathbb{R}^2 \times \mathbb{R}$ is a smooth function that can be taken to have the form

$$\varphi_U(\theta, \epsilon) = (\varphi(\theta, \epsilon), \epsilon).$$

Here the image of the function $\theta \mapsto \varphi(\theta, 0)$ is contained in Γ and $\varphi(0, 0) = \xi_0$. Now, we can view the restriction of the radial projection ρ to $\mathcal{S}$, $\rho_{\mathcal{S}}$, as the function defined by

$$\rho_{\mathcal{S}}(\theta, \epsilon) = \rho(\varphi_U(\theta, \epsilon)).$$

Since $\rho_{\mathcal{S}}(\theta, 0) = \rho(\varphi(\theta, 0), 0) \equiv 0$, the restriction of ρ, represented locally by its Taylor polynomial with remainder, has the form

$$\rho_{\mathcal{S}}(\theta, \epsilon) = \rho_1(\theta)\epsilon + O(\epsilon^2)$$

on a product neighborhood of the origin in the (θ, ϵ)-space. The first order Taylor coefficient is given by

$$\rho_1(\theta) = \frac{\partial \rho_{\mathcal{S}}}{\partial \epsilon}(\theta, 0) = d\rho(\varphi(\theta, 0), 0)(\varphi_\epsilon(\theta, 0)) + \rho_\epsilon(\varphi(\theta, 0), 0).$$

But, for $\xi \in \Gamma$, a computation similar to the computation made above for $d\tau$ shows

$$d\rho(\xi, 0)(\mathbf{f}^{\perp}(\xi)) = d\rho(\xi, 0)(\mathbf{f}(\xi)) = 0.$$

Thus, $d\rho(\xi, 0) = 0$ and

$$\rho_1(\theta) = \rho_\epsilon(\varphi(\theta, 0), 0) = M(\varphi(\theta, 0)).$$

The hypothesis $M(\xi_0) = 0$ but $dM(\xi_0)(f(\xi_0)) \neq 0$ implies $\rho_1(0) = 0$ and $\rho_1'(0) \neq 0$. Thus, there is an $\epsilon_0 > 0$ and a smooth function σ_0 defined for $|\epsilon| < \epsilon_0$, with range in the θ-space, such that $\rho_{\mathcal{S}}(\sigma_0(\epsilon), \epsilon) \equiv 0$. If we define $\sigma(\epsilon) := (\varphi(\sigma_0(\epsilon), \epsilon), \epsilon)$, then

$$\rho(\sigma(\epsilon), \epsilon) \equiv \tau(\sigma(\epsilon), \epsilon) \equiv 0$$

and we have $\delta(\sigma(\epsilon), \epsilon) \equiv 0$ as required to prove the reduction.

For the identification, the derivative of the period function can be computed directly from the Variation Lemma. In fact,

$$dT(\mathbf{f}^\perp)(\xi) = -||\mathbf{f}(\xi)||^2 \alpha(T(\xi), \mathbf{f}, \xi).$$

The proof will be complete when we identify the Melnikov function. To do this, we compute $\rho_\epsilon(\xi, 0)$ as follows:

$$\begin{aligned}\rho_\epsilon(\xi, 0) &= \left\langle \delta_\epsilon(\xi, 0), \mathbf{f}^\perp(\xi) \right\rangle \\ &= \left\langle \mathbf{P}_\epsilon(\xi, 0), \mathbf{f}^\perp(\xi) \right\rangle \\ &= \left\langle \mathbf{x}_\epsilon(m\eta, \xi, 0), \mathbf{f}^\perp(\xi) \right\rangle .\end{aligned}$$

But, $\mathbf{x}_\epsilon(t, \xi, 0)$ is the solution of the variational initial value problem

$$\dot{\mathbf{W}} = D\mathbf{f}(\phi_t(\xi))\mathbf{W} + \mathbf{g}(\phi_t(\xi), t), \quad \mathbf{W}(0) = 0.$$

By the Variation Lemma, this solution is given by

$$\mathbf{x}_\epsilon(m\eta, \xi, 0) = \beta(\xi)\mathcal{M}(\xi)\mathbf{u}_{\mathbf{f}^\perp}(\xi) \pmod{\mathbf{f}}.$$

Using the fact that $\beta(\xi) \equiv 1$ and substitution of this expression into the formula for $\rho_\epsilon(\xi, 0)$ we obtain the desired result. □

When Γ is hyperbolic, the result is similar to the last theorem only the formulas for the partial derivatives of the perturbation series are more complicated. In order to state our bifurcation theorem in this context, we again consider the system E_ϵ given by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon\mathbf{G}(\mathbf{x}, t, \epsilon), \ \mathbf{x} \in \mathbb{R}^2, \ \epsilon \in \mathbb{R}$$

where the external excitation G is periodic of period η in its second variable. We assume the unperturbed system, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, has a limit cycle Γ whose period is in resonance with the period of the forcing function. In fact, we assume the period of Γ is $m\eta/n$ for m and n relatively prime positive integers.

Before stating our theorem, we pause to recall a trivial but important fact. If the limit cycle of the unperturbed system is hyperbolic, it is structurally stable in the class of plane vector fields. When we consider the forced oscillator on the manifold $\mathbb{R}^2 \times S^1$, the three dimensional system of differential equations has, for $\epsilon = 0$, a normally hyperbolic torus corresponding to the limit cycle. The flow on this torus will be periodic or quasiperiodic depending on whether or not the resonance condition is satisfied. But, in either case, the orbits corresponding to the limit cycle will no longer be structurally stable; the stability of the limit cycle has been "transferred to the torus". In the resonant case, if we consider the appropriate iterate of the Poincaré map, we will have the Poincaré map reduced to the identity on the torus. Thus, it is natural to ask if any of the fixed points of the Poincaré map, corresponding to the points on the limit cycle, survive after perturbation as fixed points or higher order periodic points of the Poincaré map, i.e., if any periodic solutions survive as harmonic or subharmonic solutions of the forced oscillator.

For our bifurcation result, the definitions of the Poincaré section Σ, the parameterized Poincaré map $\mathbf{P}$ and the displacement function δ remain unchanged. We consider $\mathbf{f}$ and $\mathbf{G}$ fixed, with

$$\mathbf{G}(\mathbf{x}, t, \epsilon) = \mathbf{g}(\mathbf{x}, t) + \epsilon \mathbf{g}_R(\mathbf{x}, t, \epsilon),$$

and, for notational convenience, when we refer to the functions α, β, $\mathcal{M}$ and $\mathcal{N}$ with the single argument ξ, we understand that the remaining arguments are fixed at $t = m\eta$, $\mathbf{f}$ and $\mathbf{g}$. In order to obtain our bifurcation result, a correction term must be added to the subharmonic Melnikov function. In fact, the new function we require is defined by

$$\mathcal{C}(\xi) := [(1-\beta)\mathcal{N} + \alpha\mathcal{M}](\xi).$$

We call $\mathcal{C}$ the *subharmonic bifurcation function.* The next theorem applies either when Γ is hyperbolic or when, at the bifurcation point, the derivative of the transit time does not vanish.

THEOREM 4.3 (Limit Cycle Subharmonic Bifurcation Theorem). *Let E_ϵ denote the parameterized family of differential equations*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon \mathbf{g}(\mathbf{x}, t) + \epsilon^2 \mathbf{g}_R(\mathbf{x}, t, \epsilon), \ \mathbf{x} \in \mathbb{R}^2, \ \epsilon \in \mathbb{R},$$

such that E_0 has a limit cycle Γ whose period is in resonance with the η-periodic external force $\mathbf{G}(x, t, \epsilon) := g(x, t) + \epsilon \mathbf{g}_R(x, t, \epsilon)$, i. e. , there are relatively prime natural numbers m and n such that the period of Γ is $m\eta/n$. If Γ is hyperbolic and $\xi \in \Gamma$ is a simple zero of the subharmonic bifurcation function $\mathcal{C}$, i. e. , $\mathcal{C}(\xi) = 0$ and $d\mathcal{C}(\mathbf{f})(\xi) \neq 0$, then ξ is a subharmonic branch point. Also, if $\xi \in \Gamma$ is a simple zero of the subharmonic bifurcation function and if $\alpha(\xi) \neq 0$, then ξ is a subharmonic branch point.

Proof. With the projections ρ and τ defined exactly as before and with $\xi \in \Gamma$, we have

$$\begin{aligned}
d\tau(\xi, 0)(\mathbf{f}(\xi)) &= \langle (D\mathbf{P}(\xi, 0) - I)\mathbf{f}(\xi), \mathbf{f}(\xi) \rangle = 0, \\
d\rho(\xi, 0)(\mathbf{f}(\xi)) &= \langle (D\mathbf{P}(\xi, 0) - I)\mathbf{f}(\xi), \mathbf{f}^{\perp}(\xi) \rangle = 0, \\
d\tau(\xi, 0)(\mathbf{f}^{\perp}(\xi)) &= \left\langle ||\mathbf{f}||^2 \left\{ \alpha \mathbf{f} + \frac{\beta}{||\mathbf{f}||^2} \mathbf{f}^{\perp} \right\} - \mathbf{f}^{\perp}, \mathbf{f} \right\rangle (\xi) = \left[||\mathbf{f}||^4 \alpha \right] (\xi), \\
d\rho(\xi, 0)(\mathbf{f}^{\perp}(\xi)) &= \left\langle ||\mathbf{f}||^2 \left\{ \alpha \mathbf{f} + \frac{\beta}{||\mathbf{f}||^2} \mathbf{f}^{\perp} \right\} - \mathbf{f}^{\perp}, \mathbf{f}^{\perp} \right\rangle (\xi) = \left[||\mathbf{f}||^2 (\beta - 1) \right] (\xi).
\end{aligned}$$

Thus, there are two choices. If Γ is hyperbolic, then $\beta \neq 1$ and we can apply the Implicit Function Theorem to the radial projection to obtain a manifold $\mathcal{S}$ transverse to Σ such that ρ is identically zero on $\mathcal{S}$ and such that $\Gamma \subset \mathcal{S}$. If, on the other hand, $\alpha(\xi) \neq 0$, there is a manifold $\mathcal{S}$, defined locally in a neighborhood U of $(\xi, 0)$, such that τ is identically zero on $\mathcal{S}$, $\mathcal{S}$ is transverse to $\Sigma \cap U$ and $\Gamma \cap U \subset \mathcal{S}$. In both cases, the local coordinates are given by

$$(\theta, \epsilon) \mapsto (\varphi(\theta, \epsilon), \epsilon)$$

with $\varphi(\theta, 0) = \xi$. Thus, in the hyperbolic case we can restrict τ to $\mathcal{S}$ and obtain the representation

$$\tau_{\mathcal{S}}(\theta, \epsilon) := \tau_1(\theta)\epsilon + O(\epsilon^2)$$

while in case $\alpha(\xi) \neq 0$, we restrict the projection ρ to $\mathcal{S}$ to obtain

$$\rho_{\mathcal{S}}(\theta, \epsilon) = \rho_1(\theta)\epsilon + O(\epsilon^2).$$

The reduction portion of the theorem is the content of the following propositions: In the hyperbolic case, a simple zero of $\theta \to \tau_1(\theta)$ is a subharmonic branch point, while in the case $\alpha(\xi) \neq 0$, a simple zero of $\theta \to \rho_1(\theta)$ is a subharmonic branch point.

We will complete the proof by identification of the functions τ_1 and ρ_1. For this, note that

$$\begin{aligned}\tau_1(\theta) = \frac{\partial \tau_{\mathcal{S}}}{\partial \epsilon}(\theta, 0) &= d\tau(\varphi(\theta, 0), 0)(\varphi_\epsilon(\theta, 0)) + \tau_\epsilon(\varphi(\theta, 0), 0)\\ &= d\tau(\xi, 0)(\varphi_\epsilon(\theta, 0)) + \tau_\epsilon(\xi, 0)\end{aligned}$$

and

$$\begin{aligned}\rho_1(\theta) = \frac{\partial \rho_{\mathcal{S}}}{\partial \epsilon}(\theta, 0) &= d\rho(\varphi(\theta, 0), 0)(\varphi_\epsilon(\theta, 0)) + \rho_\epsilon(\varphi(\theta, 0), 0)\\ &= d\rho(\xi, 0)(\varphi_\epsilon(\theta, 0)) + \rho_\epsilon(\xi, 0).\end{aligned}$$

As in the last computation of the previous theorem, using the Variation Lemma and remembering that $\beta(\xi)$ may not be unity, we find

$$\tau_\epsilon(\xi, 0) = ||\mathbf{f}||^2(\mathcal{N} + \alpha\mathcal{M})(\xi), \qquad \rho_\epsilon(\xi, 0) = (\beta\mathcal{M})(\xi).$$

To compute the other terms we observe that $\varphi(\theta, \epsilon) \in \Sigma$. Hence, the vector field φ_ϵ can be expressed as a linear combination of $\mathbf{f}$ and $\mathbf{f}^\perp$ evaluated at $\varphi(\theta, \epsilon)$. In fact, there are scalars a and b (perhaps different in the two cases) such that

$$\varphi_\epsilon(\theta, 0) = a\mathbf{f}(\xi) + b\mathbf{f}^\perp(\xi).$$

Moreover, since both $d\tau(\xi, 0)\mathbf{f}(\xi) = 0$ and $d\rho(\xi, 0)\mathbf{f}(\xi) = 0$ we have

$$d\tau(\xi, 0)\varphi_\epsilon(\theta, 0) = b||\mathbf{f}||^4\alpha(\xi), \qquad d\rho(\xi, 0)\varphi_\epsilon(\theta, 0) = \left[b(\beta - 1)||\mathbf{f}||^2\right](\xi).$$

Now, after substitution, we obtain

$$\tau_1(\theta) = \left[b||\mathbf{f}||^4\alpha + ||\mathbf{f}||^2(\mathcal{N} + \alpha\mathcal{M})\right](\xi), \qquad \rho_1(\theta) = \left[b(\beta - 1)||\mathbf{f}||^2 + \beta\mathcal{M}\right](\xi)$$

and it suffices, in each case, to compute b. For this, we recall that on $\mathcal{S}$, in the hyperbolic case,

$$\rho(\varphi(\theta, \epsilon), \epsilon) \equiv 0$$

so

$$d\rho(\varphi(\theta, 0), 0)(\varphi_\epsilon(\theta, 0)) + \rho_\epsilon(\varphi(\theta, 0), 0) = 0$$

and we have

$$b\, d\rho(\xi, 0)\mathbf{f}^\perp(\xi) = -\rho_\epsilon(\xi, 0).$$

Thus, we can solve for b to obtain

$$b = \frac{\beta\mathcal{M}}{||\mathbf{f}||^2(1-\beta)}$$

and, after substitution into the last formula for τ_1, we compute

$$\tau_1(\theta) = \frac{||\mathbf{f}||^2}{1-\beta}\left[(1-\beta)\mathcal{N} + \alpha\mathcal{M}\right](\theta).$$

In case $\alpha(\xi) \neq 0$, the proof is the similar. We find

$$b = -\frac{\tau_\epsilon(\xi, 0)}{||\mathbf{f}(\xi)||^4\alpha(\xi)}$$

and

$$\rho_1(\theta) = \frac{1}{\alpha}\left[(1-\beta)\mathcal{N} + \alpha\mathcal{M}\right](\theta)$$

as required. □

We have just considered subharmonic bifurcation from periodic trajectories of our unperturbed system in the most degenerate case, when the unperturbed system has an isochronous center, and in the least degenerate case, when the kernel of the space derivative of the displacement function at the unperturbed periodic orbit Γ is one dimensional. A bifurcation theory for the cases of intermediate degeneracy can be carried out quite generally using our methods. However, since computation of the higher order derivatives which appear in the analysis increase in complexity, we are content to illustrate the analysis in the least degenerate of the remaining cases. For this, it should now be clear that there are two possibilities depending on whether or not Γ is a limit cycle. If Γ is not a limit cycle, it belongs to a period annulus. This means $\beta(\xi) \equiv 1$ for ξ in this period annulus and the degenerate case is $\alpha(\xi) \equiv 0$ but $d\alpha(\xi)\mathbf{f}^\perp(\xi) \neq 0$ for $\xi \in \Gamma$, i.e., the derivative of the period function vanishes on Γ but its second derivative does not vanish. This case has been treated by different methods when the unperturbed system is Hamiltonian in [38] and in a more abstract setting in [24]. In case Γ is a nonhyperbolic limit cycle, say for $\xi \in \Gamma$, $\beta(\xi) \equiv 1$ but $d\beta(\xi)\mathbf{f}^\perp(\xi) \neq 0$, we already know the bifurcation is not degenerate at a point $\xi \in \Gamma$ where $\alpha(\xi) \neq 0$. So the bifurcation is degenerate when this is not the case, i.e., when Γ is a nonhyperbolic limit cycle of multiplicity 2, cf. [1, p.272], which contains points where the derivative of the transit time map on orthogonal sections vanishes. For these cases there is the possibility that more than one family of subharmonics is found near a subharmonic branch point. We will say $\xi \in \Gamma$ is a *subharmonic branch point with n-branches* if there is an $\epsilon_0 > 0$ and distinct (germs of) curves (at $\epsilon = 0$), $\epsilon \mapsto \sigma_k(\epsilon)$; $k = 1, \ldots, n$, each defined either for $\epsilon_0 < \epsilon \leq 0$, or for $0 \leq \epsilon < \epsilon_0$, and each with image in the section Σ, such that $\sigma_k(0) = \xi$ and $\delta(\sigma_k(\epsilon), \epsilon) \equiv 0$. The next theorem gives the result for the case of the period annulus.

THEOREM 4.4 (Order 2 Subharmonic Bifurcation Theorem). *Let E_ϵ denote the parameterized family of differential equations*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon\mathbf{g}(\mathbf{x}, t) + \epsilon^2\mathbf{g}_R(\mathbf{x}, t, \epsilon),\ \mathbf{x} \in \mathbb{R}^2,\ \epsilon \in \mathbb{R},$$

such that E_0 has a period annulus $\mathcal{A}$ and a periodic trajectory $\Gamma \subset \mathcal{A}$ that is in resonance with the η-periodic external force $\mathbf{G}$, *i. e. , there are relatively prime natural* *numbers m and n such that the period of* Γ *is* $m\eta/n$*. If* Γ *is critical* $(\alpha(\xi)\equiv 0)$ *and if* $\xi\in\Gamma$ *is a simple zero of the subharmonic Melnikov function, such that*

$$\mathcal{N}(\xi)d\alpha(\xi)\mathbf{f}^{\perp}(\xi)\neq 0,$$

then ξ *is a subharmonic branch point with two branches. Moreover, these two branches exist only in the direction of* ϵ *such that*

$$\epsilon\mathcal{N}(\xi)d\alpha(\xi)\mathbf{f}^{\perp}(\xi)<0.$$

Proof. Fix $\xi\in\Gamma$, let $t\to\phi_t$ denote the flow of $\dot{x}=\mathbf{f}(x)$, let $s\to\psi_s$ denote the flow of $\dot{x}=\mathbf{f}^{\perp}(x)$, and consider the local coordinates defined at $(\xi,0)$ by the transformation

$$(s,t,\epsilon)\to\left(\psi_s\left(\phi_t(\xi)\right),\epsilon\right).$$

In these coordinates we have

$$\tau_{loc}(s,t,\epsilon):=\tau\left(\psi_s\left(\phi_t(\xi)\right),\epsilon\right).$$

However, for notational convenience, we will write τ for τ_{loc}. Using this convention, we see immediately that $\tau(0,t,0)\equiv 0$. Also, since

$$\tau_s(s,t,0)=||\mathbf{f}\left(\psi_s\left(\phi_t(\xi)\right)\right)||^4\alpha\left(\psi_s\left(\phi_t(\xi)\right)\right),$$

we have

$$\tau_s(0,t,0)=||\mathbf{f}\left(\phi_t(\xi)\right)||^4\alpha\left(\phi_t(\xi)\right)\equiv 0$$

and, for each $t\in\mathbb{R}$, we compute

$$\tau_{ss}(0,t,0)=||\mathbf{f}\left(\phi_t(\xi)\right)||^4 d\alpha\left(\phi_t(\xi)\right)\mathbf{f}^{\perp}\left(\phi_t(\xi)\right)\neq 0.$$

By an application of the (Weierstrass) Preparation Theorem,

$$\tau(s,t,\epsilon)=\left(a(t,\epsilon)+b(t,\epsilon)s+s^2\right)u(s,t,\epsilon)$$

for functions a, b and u, of the indicated variables, which satisfy

$$a(t,0)\equiv 0,\quad b(t,0)\equiv 0,\quad u(0,t,0)\neq 0.$$

In particular, there are functions $t\to a_1(t)$ and $t\to b_1(t)$ such that

$$a(t,\epsilon)=a_1(t)\epsilon+O(\epsilon^2),\qquad b(t,\epsilon)=b_1(t)\epsilon+O(\epsilon^2)$$

and we have $\tau(s(t,\epsilon),t,\epsilon)\equiv 0$ for $s(t,\epsilon)$ denoting one of the two roots

$$\frac{-b(t,\epsilon)\pm\sqrt{-4a_1(t)\epsilon+O(\epsilon^2)}}{2}$$

of the Weierstrass polynomial. Now, if $a_1(0)\neq 0$, there is a branched surface $\mathcal{S}$ with exactly two branches along Γ in the direction of ϵ such that $a_1(0)\epsilon<0$. In fact, for each root there is a locally defined "surface" given by

$$(t,\epsilon)\to(s(t,\epsilon),t,\epsilon)$$

that contains Γ and is such that $\tau(s(t,\epsilon),t,\epsilon)\equiv 0$.

To identify the quantity $a_1(0)$, we first compute

$$\tau_\epsilon(0,0,0)=a_\epsilon(0,0)u(0,0,0)=a_1(0)u(0,0,0).$$

But, in addition, we know

$$\tau_\epsilon(0,0,0)=\tau_\epsilon(\xi,0)=\left[(\mathcal{N}+\alpha\mathcal{M})||\mathbf{f}||^2\right](\xi)=||\mathbf{f}(\xi)||^2\mathcal{N}(\xi)$$

and

$$||\mathbf{f}(\xi)||^4 d\alpha(\xi)\mathbf{f}^\perp(\xi)=\tau_{ss}(0,0,0)=2u(0,0,0).$$

Thus, after substitution, we obtain

$$a_1(0)=\frac{2\mathcal{N}(\xi)}{||\mathbf{f}(\xi)||^2 d\alpha(\xi)\mathbf{f}^\perp(\xi)}$$

and we see there will be a real branched surface with two branches provided

$$\epsilon\mathcal{N}(\xi)d\alpha(\xi)\mathbf{f}^\perp(\xi)<0.$$

Next, as before, we consider the restriction of the projection ρ to $\mathcal{S}$. However, here there is a slight difference from our previous arguments because the function s is not necessarily smooth at $\epsilon=0$. To overcome this difficulty we must incorporate the Puiseux series for our expansion of s in powers of ϵ. In the quadratic case this is quite simple. In fact, under our hypothesis that $a_1(0)\neq 0$, there exits a function $(t,\zeta)\to s^*(t,\zeta)$, analytic at $(0,0)$, such that

$$s(t,\epsilon)=s^*(t,\sqrt{\epsilon}).$$

When we restrict the projection ρ to the corresponding branch of $\mathcal{S}$, we have a function $(t,\zeta)\to\rho^*(t,\zeta)$, analytic at $(0,0)$, defined by

$$\rho^*(t,\zeta):=\rho_{loc}(s^*(t,\zeta),t,\zeta^2)$$

with

$$\rho_{\mathcal{S}}(t,\epsilon)=\rho^*(t,\sqrt{\epsilon}).$$

Now, using the definition of ρ,

$$\rho_{loc}(s,t,\epsilon):=\rho\left(\psi_s\left(\phi_t(\xi)\right),\epsilon\right),$$

but henceforth writing ρ for ρ_{loc}, we obtain from previous computations, $\rho(0,t,0)\equiv 0$, and

$$\rho_s(s,t,0)=d\rho(s,t,0)\mathbf{f}^\perp(s,t)=-||\mathbf{f}(s,t)||^2(1-\beta(s,t))\equiv 0.$$

A calculation using these facts and the chain rule yields $\rho^*_\zeta(t,0)\equiv 0$ and

$$\rho^*_{\zeta\zeta}(t,0)=2\rho_\epsilon(0,t,0)=2\beta\left(\phi_t(\xi)\right)\mathcal{M}\left(\phi_t(\xi)\right).$$

Thus, we have the representation

$$\rho^*(t,\zeta)=\rho_2(t)\zeta^2+O(\zeta^3)$$

with $\rho_2(t) = \mathcal{M}\left(\phi_t(\xi)\right)$ and we see that if ξ is a simple zero of the Melnikov function, then $t = 0$ is a simple zero of ρ_2 and the Implicit Function Theorem applies to show the existence of a curve $\zeta \to \sigma(\zeta)$, analytic at $\zeta = 0$, and such that $\rho^*(\sigma(\zeta), \zeta) \equiv 0$. It follows that

$$\rho_{\mathcal{S}}(\sigma(\sqrt{\epsilon}), \epsilon) = \rho^*\left(\sigma(\sqrt{\epsilon}), \sqrt{\epsilon}\right) \equiv 0$$

and therefore $t = \sigma(\sqrt{\epsilon})$, i.e.,

$$\epsilon \to \psi_{s(\sigma(\sqrt{\epsilon}),\epsilon)}\left(\phi_{\sigma(\sqrt{\epsilon})}(\xi)\right)$$

is the desired branch of subharmonics at $\xi \in \Gamma$. □

THEOREM 4.5 (Order 2 Limit Cycle Subharmonic Bifurcation Theorem). *Let E_ϵ denote the parameterized family of differential equations*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon \mathbf{g}(\mathbf{x}, t) + \epsilon^2 \mathbf{g}_R(\mathbf{x}, t, \epsilon), \ \mathbf{x} \in \mathbb{R}^2, \ \epsilon \in \mathbb{R},$$

such that E_0 has a periodic trajectory $\Gamma \subset \mathcal{A}$ that is in resonance with the η-periodic external force $\mathbf{G}$, i. e. , there are relatively prime natural numbers m and n such that the period of Γ is $m\eta/n$. If $\xi \in \Gamma$ is such that following three conditions are satisfied: (i) $\alpha(\xi) = 0$ and $\beta(\xi) = 1$, (ii) either $\mathcal{M}(\xi) d\beta(\xi)\mathbf{f}^\perp(\xi) \neq 0$ or $\mathcal{N}(\xi) d\alpha(\xi)\mathbf{f}^\perp(\xi) \neq 0$, and (iii) $\xi \in \Gamma$ is a simple zero of the bifurcation function

$$\mathcal{D} := \mathcal{N}(\xi) d\beta(\xi)\mathbf{f}^\perp(\xi) - \mathcal{M}(\xi) d\alpha(\xi)\mathbf{f}^\perp(\xi),$$

then ξ is a subharmonic branch point with two branches. Moreover, these two branches exist, in case $\mathcal{M}(\xi) d\beta(\xi)\mathbf{f}^\perp(\xi) \neq 0$, only in the direction of ϵ such that

$$\epsilon \mathcal{M}(\xi) d\beta(\xi)\mathbf{f}^\perp(\xi) < 0$$

and, in case $\mathcal{N}(\xi) d\alpha(\xi)\mathbf{f}^\perp(\xi) \neq 0$, only in the direction of ϵ such that

$$\epsilon \mathcal{N}(\xi) d\alpha(\xi)\mathbf{f}^\perp(\xi) < 0.$$

Proof. The proof of this theorem follows exactly the same logic as the proof of the last theorem with only a few complications. The Preparation Theorem is applied in turn to both projections ρ and τ, but the proof in both cases is the same. Also, using the notation developed in the proof of the last theorem, we must deal with the fact that neither $\rho_{ss}(0,0,0)$ nor $\tau_{ss}(0,0,0)$ must vanish. For example, in the computation of $\tau^*_{\zeta\zeta}$ we obtain

$$\tau^*_{\zeta\zeta}(0,0) = \tau_{ss}(0,0,0)[s^*_\zeta(0,0)]^2 + 2\tau_\epsilon(0,0,0).$$

So, we must compute $s^*_\zeta(0,0)$. However, using the quadratic formula and the definition of s^* it is clear that

$$[s^*_\zeta(0,0)]^2 = -a_1(0),$$

with the quantity $a_1(0)$ computable as before. Thus, by similar, but slightly more complicated computations, the theorem can be proved by computation of the bifurcation derivatives in terms of α, β, $\mathcal{M}$, and $\mathcal{N}$. □

We end this section with an important remark on detuning. For this, consider a parametrized family of differential equations E_ϵ given by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon \mathbf{G}(\mathbf{x}, t, \epsilon), \quad \mathbf{x} \in \mathbb{R}^2, \ \epsilon \in \mathbb{R},$$

where E_0 has a periodic trajectory Γ. Up to now we have assumed the period η of the excitation $t \to \mathbf{G}(\mathbf{x}, t, \epsilon)$ is in resonance with the period of Γ. However, this condition can easily be relaxed in our analysis. In fact, we can assume merely that the period of the excitation is given by an expression of the form $\eta + k\epsilon + O(\epsilon^2)$. In this situation the parameter $k \in \mathbb{R}$ is called a *detuning.* When detuning is introduced, we retain our resonance assumption that the period of Γ is equal to $m\eta/n$ and simply reformulate the analysis in terms of an appropriate Poincaré map, namely

$$\mathbf{P}(\xi, \epsilon) = \mathbf{x}(m\eta + mk\epsilon + O(\epsilon^2), \xi, \epsilon),$$

where $\mathbf{x}(t, \xi, \epsilon)$ is the solution of E_ϵ with $\mathbf{x}(0, \xi, \epsilon) = \xi$. Clearly, all the derivatives of $\mathbf{P}$ with respect to the space variable ξ reduce to previously computed expressions when evaluated at $\epsilon = 0$. On the other hand, we have

$$\begin{aligned}
\mathbf{P}_\epsilon(\xi, 0) &= \dot{\mathbf{x}}(m\eta, \xi, 0)mk + \mathbf{x}_\epsilon(m\eta, \xi, 0) \\
&= mk\mathbf{f}(\xi) + \left[(\mathcal{N} + \alpha\mathcal{M})\mathbf{f} + \beta\mathcal{M}\mathbf{u}_{\mathbf{f}^\perp}\right](\xi) \\
&= \left[((mk + \mathcal{N}) + \alpha\mathcal{M})\mathbf{f} + \frac{1}{||\mathbf{f}||^2}\beta\mathcal{M}\mathbf{f}^\perp\right](\xi).
\end{aligned}$$

Thus, all previous statements of theorems and formulas for derivatives remain valid in the case of a detuning when we replace each occurrence of $\mathcal{N}$ with $mk + \mathcal{N}$.

5. Examples. As a first example to illustrate the use of the Limit Cycle Subharmonic Bifurcation Theorem, we consider the nonlinear system given by

$$\dot{x} = -y + x(1 - x^2 - y^2) + \epsilon g_1(x, y, t), \qquad \dot{y} = x + y(1 - x^2 - y^2) + \epsilon g_2(x, y, t)$$

where $G(x, y, t) := (g_1(x, y, t), g_2(x, y, t))$ is t-periodic of period $\eta := 2\pi n/m$ for some relatively prime positive integers n and m. Also, we denote the associated vector field of the unperturbed system by $\mathbf{x}$. This vector field is chosen to have a simple representation in polar coordinates,

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = 1$$

and a unique hyperbolic limit cycle Γ of period 2π on the unit circle. In fact the integral curve of $\mathbf{x}$ corresponding to Γ and starting at $\xi := (\xi_1, \xi_2)$ with $|\xi| = 1$ is given by

$$x(t) = \xi_1 \cos t - \xi_2 \sin t, \quad y(t) = \xi_1 \sin t + \xi_2 \cos t.$$

For this example, we compute $\alpha(t) \equiv 0$ on Γ. Thus, the bifurcation function is

$$\mathcal{C}(\xi) = \left[(1 - \beta)\mathcal{N}\right](\xi) = (1 - e^{-2m\eta}) \int_0^{m\eta} x g_2(x, y, t) - y g_1(x, y, t)\, dt.$$

If we specify the forcing function, we can now determine the existence of subharmonic branch points. For example, if

$$g_1(t) = a \cos t + b \sin t, \qquad g_2(t) = c \cos t + d \sin t$$

we compute

$$\mathcal{C}(\xi) = n\pi \left(1 - e^{-4n\pi}\right) ((c-b)\xi_1 - (a+d)\xi_2)$$

and we see there are generically two subharmonic branch points at the intersection of the unit circle with the line $(c-b)\xi_1 - (a+d)\xi_2 = 0$. For an example where the excitation depends on the space variables, we take

$$g_1(x,y,t) = ax\cos\frac{t}{2}, \qquad g_2(x,y,t) = by\sin\frac{t}{2}$$

and compute, taking $n = 1$,

$$\mathcal{C}(\xi) = \frac{8}{15}(b-a)\left(1 - e^{-4\pi}\right)(\xi_1 - \xi_2)(\xi_1 + \xi_2).$$

Thus, there are four subharmonic branch points corresponding to the intersections of the lines

$$\xi_1 - \xi_2 = 0, \qquad \xi_1 + \xi_2 = 0$$

with the unit circle.

In the above example we are able to give a complete mathematical analysis of the subharmonic response of a system whose free oscillation is a limit cycle when the system is subjected to a resonant periodic external excitation. We are not able at present to give a similar rigorous mathematical analysis for a model equation which arises from a physical problem. However, we have been successful in applying the theory using numerical experiments. To illustrate this, we consider the example mentioned in the introduction of a forced van der Pol oscillator. In fact, we consider the system

$$\begin{aligned}
\dot{u} &= v \\
\dot{v} &= -u + \delta(1-u^2)v \\
\dot{x} &= \tau y \\
\dot{y} &= \tau(-x + \delta(1-x^2)y) + \epsilon u,
\end{aligned}$$

where τ, δ and ϵ are real parameters. Here, we view

$$\dot{x} = \tau y, \qquad \dot{y} = \tau(-x + \delta(1-x^2)y)$$

as the unperturbed system. It has, for $\delta > 0$ and $\tau > 0$, a stable limit cycle as its free oscillation. If, in addition, τ is a rational number and $\epsilon \neq 0$, then the xy-system is perturbed by a periodic external stimulus provided by the periodic output $t \mapsto u(t)$ of the uv-system, a second van der Pol oscillator running in resonance. To find the number and the positions of the subharmonic branch points where, for sufficiently small ϵ, families of subharmonic solutions of the perturbed oscillator emerge, we must find the simple zeros of the subharmonic bifurcation function $\mathcal{C}$ along the unperturbed limit cycle. Here, we are restricted by the lack of explicit analytic expressions for the solutions of the unperturbed system near its stable limit cycle. Thus, we have resorted to numerical experiments in order to suggest the actual subharmonic response. One can compute the graph of the bifurcation function $\mathcal{C}$ numerically to obtain the subharmonic branch points for various choices of the parameters. A typical graph of this type is depicted in Figure 1 of the introduction.

Our final example is an application of the Order 2 Limit Cycle Subharmonic Bifurcation Theorem. For this, consider the system E_ϵ given by

$$\dot{x} = y - x(1 - x^2 - y^2)^2 - \epsilon \cos t, \qquad \dot{y} = -x - y(1 - x^2 - y^2)^2 + \epsilon \sin t.$$

which has the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon \mathbf{g}(t).$$

Here, the unperturbed system has a (semi-stable) multiplicity 2 limit cycle Γ on the unit circle. The corresponding integral curve of E_0 starting at $\xi := (\xi_1, \xi_2)$ is

$$x(t) = \xi_1 \cos t + \xi_2 \sin t, \quad y(t) = -\xi_1 \sin t + \xi_2 \cos t.$$

To apply the theorem, we define $r := \sqrt{x^2 + y^2}$, and compute

$$\begin{aligned} \operatorname{div} \mathbf{f}(x,y) &= -2(r^2 - 1)(3r^2 - 1) \\ \operatorname{curl} \mathbf{f}(x,y) &= -2 \\ 2\kappa ||\mathbf{f}||(x,y) &= -2 \frac{2 - 6r^4 + 8r^6 - 3r^8}{2 - 4r^2 + 6r^4 - 4r^6 + r^8}. \end{aligned}$$

Then, for $\xi \in \Gamma$, we can compute

$$\begin{aligned} &\alpha(t,\xi) \equiv 0, \quad && d\alpha(\xi)\mathbf{f}^\perp(\xi) = 0, \\ &\beta(\xi) = 1, && d\beta(\xi)\mathbf{f}^\perp(\xi) = -16\pi. \end{aligned}$$

For example, we have $\mathbf{f}^\perp(\xi) = \xi$ and, for any $\zeta \in \mathbb{R}^2$,

$$\beta(\zeta) = \exp\left(\int_0^{2\pi} -2(r^2 - 1)(3r^2 - 1)\, dt \right).$$

Thus,

$$d\beta(\xi)\mathbf{f}^\perp(\xi) = d\beta(\xi)\xi = \left. \frac{d}{ds} \beta((1+s)\xi) \right|_{s=0}.$$

After the obvious computation, we find

$$d\beta(\xi)\mathbf{f}^\perp(\xi) = -8 \int_0^{2\pi} \left. \frac{dr}{ds} \right|_{s=0} dt.$$

But, since

$$\dot{r} = -r^2(1 - r^2)^2,$$

we can easily find the variational equation for r_s on Γ to be

$$\dot{r}_s = 0.$$

Also, since $r(0,s) = ||(1+s)\xi||$, we have $r_s(0,0) = 1$. This means $dr/ds \equiv 1$ and, in turn,

$$d\beta(\xi)\mathbf{f}^\perp(\xi) = -16\pi.$$

Next, we find

$$\mathcal{M}(\xi) = \int_0^{2\pi} y \sin t - x \cos t \, dt = -2\pi\xi_1.$$

Thus, the only zeros of $\mathcal{M}(\xi) = 0$ are on the line $\xi_1 = 0$. In particular, for $\zeta = (\pm 1, 0)$,

$$\mathcal{M}(\zeta) d\beta(\zeta) \mathbf{f}^{\perp}(\zeta) \neq 0.$$

We also have

$$\mathcal{D}(\xi) = -16\pi\mathcal{N}(\zeta) = 16\pi \int_0^{2\pi} (y \cos t + x \sin t) \, dt = 32\pi^2 \xi_2.$$

Thus, $\mathcal{D}$ has a simple zero along Γ at ζ and, by the theorem, there will be two branches of harmonics at the subharmonic branch point ζ. At $\zeta = (-1, 0)$ these harmonics exist for sufficiently small $\epsilon < 0$ while at $(1, 0)$ they exist for sufficiently small $\epsilon > 0$.

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