

Math 426 Final: Due Wed. Dec 17 before
2:00P.M. (No Extensions!)

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December 16, 2003

1. The system $\dot{x} = x - y + 2 \sin(x)$, $\dot{y} = x + y + x^2 y$ has a rest point at the origin. Draw (with explanation) the phase portrait of this system near the origin. Determine the stability type at the origin.
2. Determine the phase portrait of the ODE $\ddot{x} + \dot{x}(x^2 + \dot{x}^2 - 1) + x = 0$.
3. The linearized Hill's equations for the relative motion of two satellites with respect to a circular reference orbit about the earth are given by

$$\begin{aligned}\ddot{x} - 2n\dot{y} - 3n^2 x &= 0, \\ \ddot{y} + 2n\dot{x} &= 0, \\ \ddot{z} + n^2 z &= 0\end{aligned}$$

where n is a positive constant related to the radius of the reference orbit and the gravitational constant. Find the flow of this system. There is a five-dimensional invariant manifold in the six-dimensional phase space corresponding to periodic orbits. An orbit with an initial condition not on this manifold contains a secular drift term. Determine the manifold of periodic orbits and explain what is meant by a secular drift term. Answer: The manifold of periodic orbits is the hyperplane given by $\dot{y} + 2nx = 0$.

4. Do exercise 2.45, page 161 of the text book.
5. Consider the general 2×2 linear system with constant coefficients and show that the system decouples in polar coordinates. The first-order differential equation for the angular coordinate θ can be viewed as a

differential equation on the unit circle \mathbb{T}^1 . Why? Consider the first-order differential equation

$$\dot{\theta} = \alpha \cos^2 \theta + \beta \cos \theta \sin \theta + \gamma \sin^2 \theta.$$

If $4\alpha\gamma - \beta^2 > 0$, prove that all orbits on the circle are periodic with period $4\pi(4\alpha\gamma - \beta^2)^{-1/2}$. Use this result to determine the period of the periodic orbits of the differential equation $\dot{\theta} = \eta + \cos \theta \sin \theta$ as a function of the parameter $\eta > 1$. Describe the behavior of this function as $\eta \rightarrow 1^+$ and give a qualitative explanation of the behavior. Repeat the last part of the exercise for the differential equation $\dot{\theta} = \eta - \sin \theta$ where $\eta > 1$.

FYI only. You do not have to hand in anything beyond this point unless you get excited. If you do get excited, I will be happy to discuss your work. An n -dimensional homogeneous linear differential equation induces a differential equation on the real projective space of dimension $n - 1$. There is an intimate connection between the linear second-order differential equation

$$\ddot{y} - (q(t) + \dot{p}(t)/p(t))\dot{y} + r(t)p(t)y = 0$$

and the Riccati equation

$$\dot{x} = p(t)x^2 + q(t)x + r(t).$$

In fact, these equations are related by $x = -\dot{y}/(p(t)y)$. For example $\ddot{y} + y = 0$ is related to the Riccati equation $\dot{u} = -1 - u^2$, where in this case the change of variables is $x = \dot{y}/y$. Also, note that the equation given by

$$\dot{x} = x(1 - x) + h(t)$$

is a Riccati equation. Note that the unit circle in \mathbb{R}^2 , with coordinates (y, \dot{y}) , has coordinate charts given by $(y, \dot{y}) \mapsto \dot{y}/y$ and $(y, \dot{y}) \mapsto y/\dot{y}$. Thus, the transformation from the linear second-order equation to the Riccati equation is a local coordinate representation of the differential equation induced by the second-order linear differential equation on the circle. Explore and explain the relation between this coordinate representation and the polar coordinate representation of the first-order linear system given in the first part of the exercise.