

# Math 426 Homework 1

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Hand in problems 5,7,9,13 on Sept. 1.

Prepare the rest of the problems to present to the class starting Aug. 27.

General rules: Please do not consult books where you can find the answers to exercises. Also, don't ask other professors in the math department for help. You can collaborate with another student, but always use your own words in the assignments you hand in for credit.

Someday soon you may wish to start publishing your work. To do so, you will have to learn to write in complete sentences using well-constructed English prose. So, please write complete sentences. Also, please write neatly so that your professor can read your work without too much difficulty.

1. Find the general solution of the ODE  $\dot{x} = x/t + t$  for  $t > 0$ . Generalize to  $\dot{x} = a(t)x + b(t)$ .
2. Solve the ODE  $\ddot{x} + \dot{x} + 10x = 0$ .
3. Solve the ODE  $dx/dt = t/(x + 1)$ .
4. Find the general solution of the ODE  $\dot{x} = x(1 - x)$ . Suppose that  $x(0) = 1/2$  what is  $\lim_{t \rightarrow \infty} x(t)$ .
5. A massive ball dropped into water sinks according to the formula

$$m\ddot{x} = mg - k\dot{x}^2,$$

where  $x$  is the depth of the ball at time  $t$ ,  $m$  is its mass,  $g$  is the gravitational constant, and  $k$  is a positive parameter that depends on the viscosity of water and the size of the ball. (1) Find the terminal velocity of the ball. I found out that in Olympic two-man bobsled

there is a weight limit of 390 kilograms, including the driver and the brakeman. Teams are allowed to add ballast as long as they stay under this weight. (2) Why would they want to add ballast? (3) Suppose that  $\dot{x} = v$ ,  $m = 1$ ,  $g = 32$ ,  $k = 1/2$ ,  $x(0) = 0$ , and  $v(0) = 0$ . Find  $x(10)$ .

6. Find the general solution of the system

$$\dot{x} = y, \quad \dot{y} = -x.$$

7. Find the general solution of the system

$$\dot{x} = -2x + 4y, \quad \dot{y} = x + y.$$

8. Consider the initial value problem

$$\ddot{x} + 2\dot{x} + 2x = 3 \cos t, \quad x(0) = 1, \quad \dot{x}(0) = 0.$$

Describe the behavior of the solution after a long time has passed, that is, after the independent variable  $t$  is large.

9. Is the uniform limit of a sequence of real continuous functions of a real variable continuous? Prove a theorem or give a counterexample.
10. Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  denote the function given by

$$x \mapsto \langle x, x \rangle x$$

where the angle brackets denote an inner product on  $\mathbb{R}^2$ . Find the derivative of  $f$ . Find the directional derivative of  $f$  at the point  $x = (1, 2)$  in the direction of the vector  $v = (3, 5)$ .

11. Prove: The continuous image of a compact set is compact.
12. Prove: If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous and  $K \subset \mathbb{R}^n$  is compact, then  $f(K)$  is bounded. Moreover, there is a point  $p \in K$  such that  $f(p)$  is the least upper bound of the set  $f(K)$ .
13. Prove: If  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a class  $C^1$  function and  $K$  is a compact subset of  $\mathbb{R}^n$ , then there is a constant  $\lambda > 0$  such that  $|f(x) - f(y)| \leq \lambda|x - y|$  whenever  $x, y \in K$ .