Math 426 Homework 2

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1. Construct infinitely many different solutions of the initial value problem

$$\dot{x} = x^{1/3}, \qquad x(0) = 0.$$

   Why does the Existence and Uniqueness Theorem for differential equations fail to apply in this case?

2. Prove that every solution of the differential equation $\dot{x} = \sin^2(x)$ exists for all time; that is for every $t \in (-\infty, \infty)$. What about solutions of $\dot{x} = x^4/(x^2 + 1)$?

3. There is a famous story about someone who wrote a thesis about the nice properties of Hölder functions with Hölder constants larger than one. A function f such that $|f(x) - f(y)| \le L|x - y|^\alpha$ for all x and y, where $L \ge 0$ and $0 \le \alpha \le 1$ are constants, is called a Hölder function with Hölder constant α. Such a function is called Lipschitz (as we know) in case $\alpha = 1$. What did the student learn (the hard way) at his thesis defense? That is, why is the class of "Hölder functions" with Hölder constants $\alpha > 1$ uninteresting?

4. (1) Suppose that $\eta : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz function with Lipschitz constant α and $0 \le \alpha < 1$. Prove that the function $F : \mathbb{R}^n \to \mathbb{R}^n$ given by $F(x) = x + \eta(x)$ is bijective. Hint: Use a contraction argument. (2) Prove that F is a homeomorphism; that is, F has a continuous inverse. Hint: Here are some possible approaches. The more abstract approaches are more difficult to execute, but they might teach you something. (a) By quoting a famous theorem, you can conclude that F has a continuous inverse. What theorem is it? (b) First prove that

F is proper; that is, the inverse image under F of each compact subset of $\mathbb{R}^n$ is compact. Then prove that a continuous bijective proper map on $\mathbb{R}^n$ has a continuous inverse. (c) Prove that the inverse of F, which exists because F is bijective, is Lipschitz and therefore continuous.