Math 426 Homework 4

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1. Prove that the phase portrait of the 2nd-order ODE $\ddot{x} - x + x^2 = 0$ has a homoclinic orbit.

2. Let $H(x, y, z) = xyz$. Does $X_H := \operatorname{grad} H - 3H\eta$, where η denotes the outer unit normal on the unit sphere, have an invariant great circle on the unit sphere? Justify your answer.

The following are some extra credit problems to contemplate. The first part of the first problem below appeared on the 1953 Putnam Exam. The problem is not too difficult. Give it a try.

1. Suppose that n is an integer. Solve the initial value problem

$$\dot{x} = y(x + y)^n, \quad \dot{y} = x(x + y)^n, \quad x(0) = 1, \quad y(0) = 0.$$

   Is the solution complete?

2. In the context of the second assigned problem, is there a general result that can be used to determine whether or not an arbitrary X_H has an invariant great circle? (I don't know how to answer this question. But, this should not stop you from trying. It might be easy.)

3. The Dinner Problem is still open (see the link in my web site devoted to your textbook). Even if you can't solve this problem, perhaps you can say something interesting. If so, I will be happy to look at your result. Restatement of the Dinner Problem: Let $\mathcal{H}$ denote the space of homogeneous cubic polynomials in three variables and let $\mathcal{A}$ denote the subset of $\mathcal{H}$ consisting of those elements whose gradients orthogonally project to a vector field on the unit sphere such that the corresponding

dynamical system has no saddle connections. Prove or disprove that $\mathcal{A}$ is a dense subset of $\mathcal{H}$ in the coefficient topology (see Exercise 1.73 in Ordinary Differential Equations with Applications).