Math 426 Homework 5

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1. Draw the phase portrait of Newtonian system $\ddot{x} = x - x^3$. Give a qualitative description of the solution of the system for the initial condition $(x(0), \dot{x}(0)) = (1, 1/\sqrt{2})$. How would the motion change qualitatively if instead the initial condition were $(x(0), \dot{x}(0)) = (1, 1/\sqrt{2} + \epsilon)$, where $|\epsilon|$ is small.

2. Prove that
$$\lim_{k \to 1^-} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \phi}}\, d\phi = \infty.$$
Recall that this proves the period function $T(k) = 4K(k)/\sqrt{\lambda}$ converges to infinity as the corresponding periodic orbits limit to the heteroclinic orbit in the phase portrait of the pendulum.

3. Consider the system
$$\dot{x} = \epsilon f(x, y), \qquad \dot{y} = g(x, y),$$
where $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$ and ϵ is a parameter. Prove: If $\epsilon = 0$ and the function $y \mapsto g(x, y)$ has an invertible derivative for every $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, then the system (for $\epsilon = 0$) has an invariant m-dimensional submanifold of $\mathbb{R}^m \times \mathbb{R}^n$. (There is a natural, interesting, and important question: Does the system have an invariant manifold when $|\epsilon|$ is not zero but small? The answer is yes. But, we have not yet discussed the tools needed for the standard proof.)