Math 426 Homework 6

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1. Find an explicit formula for the flow of the differential equaton

$$\dot{x} = -y + x(1 - x^2 - y^2), \quad \dot{y} = x + y(1 - x^2 - y^2).$$

   Show that all orbits except the rest point at the origin are asymptotic to a limit cycle.

2. Suppose that the system $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ has a periodic orbit Γ with period T and B is a positive real valued function defined on some open neighborhood of Γ (as in Dulac's Criterion). Prove that Γ is a periodic orbit of the system $\dot{x} = B(x, y)f(x, y)$, $\dot{y} = B(x, y)g(x, y)$ with period

$$\tau = \int_0^T \frac{1}{B(x(t), y(t))}\, ds$$

   where $t \mapsto (x(t), y(t))$ is a periodic solution of the original system whose orbit is Γ. How does the period of the limit cycle of system

$$\dot{x} = -y + x(1 - x^2 - y^2), \quad \dot{y} = x + y(1 - x^2 - y^2)$$

   change if its vector field is multiplied by $(1 + x^2 + y^2)^\alpha$? Hint: The solution ρ of the initial value problem

$$\dot{\rho} = B((x(\rho), y(\rho)), \quad \rho(0) = 0$$

   satisfies the identity $\rho(t + \tau) = \rho(t) + T$.

3. Suppose that $h : \mathbb{R} \to \mathbb{R}$ is a T-periodic function, and $0 < h(t) < 1/4$ for every $t \in \mathbb{R}$. Show that the differential equation $\dot{x} = x(1 - x) - h(t)$ has exactly two T-periodic solutions. The differential equation can be interpreted as a model for the growth of a population in a limiting environment that is subjected to periodic harvesting.