# Math 426 Homework 8

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October 27, 2002

1. 2.2

2. The linearized Hill's equations for the relative motion of two satellites with respect to a circular reference orbit about the earth are given by

$$\begin{aligned} \ddot{x} - 2n\dot{y} - 3n^2 x &= 0, \\ \ddot{y} + 2n\dot{x} &= 0, \\ \ddot{z} + n^2 z &= 0 \end{aligned}$$

   where $n$ is a constant related to the radius of the reference orbit and the gravitational constant. There is a five-dimensional manifold in the phase space corresponding to periodic orbits. An orbit with an initial condition not on this manifold contains a secular drift term. Determine the manifold of periodic orbits and explain what is meant by a secular drift term. Answer: The manifold of periodic orbits is the hyperplane given by $\dot{y} + 2nx = 0$.

3. Consider the general $2 \times 2$ linear system with constant coefficients and show that the system decouples in polar coordinates. The first-order differential equation for the angular coordinate $\theta$ can be viewed as a differential equation on the unit circle $\mathbb{T}^1$. Why? Consider the first-order differential equation

$$\dot{\theta} = \alpha \cos^2 \theta + \beta \cos\theta \sin\theta + \gamma \sin^2 \theta.$$

   If $4\alpha\gamma - \beta^2 > 0$, prove that all orbits on the circle are periodic with period $4\pi(4\alpha\gamma - \beta^2)^{-1/2}$. Use this result to determine the period of the periodic orbits of the differential equation $\dot{\theta} = \eta + \cos\theta \sin\theta$ as a function of the parameter $\eta > 1$. Describe the behavior of this function as $\eta \to 1^+$ and give a qualitative explanation of the behavior. Repeat the last part of the exercise for the differential equation $\dot{\theta} = \eta - \sin\theta$ where $\eta > 1$.

FYI only. You do not have to hand in anything beyond this point unless you get excited. If you do get excited, I will be happy to discuss your work. An $n$-dimensional homogeneous linear differential equation induces a differential equation on the real projective space of dimension $n-1$. There is an intimate connection between the linear second-order differential equation

$$\ddot{y} - (q(t) + \dot{p}(t)/p(t))\dot{y} + r(t)p(t)y = 0$$

and the Riccati equation

$$\dot{x} = p(t)x^2 + q(t)x + r(t).$$

In fact, these equations are related by $x = -\dot{y}/(p(t)y)$. For example $\ddot{y} + y = 0$ is related to the Riccati equation $\dot{u} = -1 - u^2$, where in this case the change of variables is $x = \dot{y}/y$. Also, note that the equation given by

$$\dot{x} = x(1-x) + h(t)$$

is a Riccati equation. Note that the unit circle in $\mathbb{R}^2$, with coordinates $(y, \dot{y})$, has coordinate charts given by $(y, \dot{y}) \mapsto \dot{y}/y$ and $(y, \dot{y}) \mapsto y/\dot{y}$. Thus, the transformation from the linear second-order equation to the Riccati equation is a local coordinate representation of the differential equation induced by the second-order linear differential equation on the circle. Explore and explain the relation between this coordinate representation and the polar coordinate representation of the first-order linear system given in the first part of the exercise.