

# Math 4560 Homework for Chapter 2

C. Chicone

Due Thursday September 3

**Very Important:** For the entire semester you must write your answers in complete sentences. This is good practice for you in scientific writing and also makes it easier for me to read (and give you a high grade). If you do not have time to do some of the assignment or a particular part of a problem, don't worry too much. But, do a good job on explaining the problems you do hand in. One other warning: Don't turn in pages of computer output with a few hand written notes in the margin. I am not interested in reading your code. Rather, for problems that require computation, explain in words what you did and illustrate your results with tables and graphs. Never present a graph in your writing without a caption that explains what the reader (me for this class) is supposed to see. One or two good graphs that get to the heart of the matter are worth more than dozens of graphs unaccompanied by clear explanations. My desire is for you to learn something from the assignments. I hope you enjoy working on at least some of the problem assigned as homework. Good luck!

1. Write up solutions for problems 2.2.11, 2.2.12, 2.2.13, 2.4.9 in Strogatz. Also, for 2.2.13, answer the following question: I found out that in Olympia two-man bobsled there is a weight limit of 390 kilograms, including the driver and the brakeman. Teams are allowed to add ballast as long as they stay under this weight. Why would they want to add ballast?
2. Do Exercise 2.8.5 about calibrating the Runge-Kutta method. As part of the exercise, make sure you test the statement: The Runge-Kutta method is a 4th order method.

3. As long as you have written an RK4 program, let's modify it and use it to approximate some solutions of an important second-order ODE called Duffing's equation (which comes up as a model in nonlinear beam theory):

$$\ddot{x} = x - x^3.$$

The first thing to do is write the second-order ODE as an equivalent system of first-order ODEs. The idea is very simple: define a new variable  $v = \dot{x}$ , which is the velocity, and note that we can now write

$$\dot{x} = v, \quad \dot{v} = x - x^3. \quad (1)$$

This is the desired second-order system. The RK4 method for systems is exactly the same as the method for scalar equations, only we vectorize everything:

$$\begin{aligned} k_1 &= \Delta t f(x_n), \\ k_2 &= \Delta t f(x_n + \frac{1}{2}k_1), \\ k_3 &= \Delta t f(x_n + \frac{1}{2}k_2), \\ k_4 &= \Delta t f(x_n + k_3), \\ x_{n+1} &= x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4). \end{aligned}$$

For the vector case, each equation is a vector equation and  $x$  is a vector, which is not the same  $x$  as in our equation. For example, for our second-order example, each equation in the RK4 equations become two equations, one for each component of  $f$ , where for us

$$f(x, v) = (v, x - x^3).$$

Thus, we will have, for example,

$$k_1^1 = \Delta t v_n, \quad k_1^2 = \Delta t(x_n - x_n^3).$$

You should test out your program on a simple example that you can solve by hand, for example,

$$\dot{x} = y, \quad \dot{y} = -x,$$

which is equivalent to the “harmonic oscillator”  $\ddot{x} + x = 0$ . The RK4 method should give a close approximation to the known solutions of this equation.

Now for the problem:

Approximate the solution of (1) for the two sets of initial data:

$$x(0) = 1, \quad \dot{x}(0) = 0.6,$$

$$x(0) = 1, \quad \dot{x}(0) = 0.8.$$

Look at the graphs of  $x$  versus  $t$  (or,  $v$  versus  $x$ ) and describe what you see. The two sets of data (if correctly integrated) will produce different results. What is the critical value of  $\dot{x}(0)$  that separates the two types of behavior? I’d like to know the correct first eight digits of this critical number. What are these digits?

Note: I did not say how to choose  $\Delta t$  and I did not specify the total time of integration. You will have to experiment! Be sure to explain which (numerical) experiments you performed. Also, explain your choice of  $\Delta t$  and the total integration time.