

MATH 4950/Winter Semester 2008/Professor C. Chicone
Capstone Homework Problems

1. Five people spend a day gathering coconuts. They bring all them back to camp, put them all in one pile and agree to divide the coconuts in the morning. During the night one of the people decides they want their share. This person is honest. She divides the coconuts into five equal piles but there is exactly one coconut left over (which she feeds to the camp monkey). She takes her share, repiles the remaining coconuts, and goes back to her tent. Later in the night each of the other four people do exactly the same thing. In the morning the camp monkey is very sick and the pile of coconuts is greatly reduced. What is the least number of coconuts that could have been gathered?
2. Get a basketball and stretch a wire around the ball. Add exactly one foot of wire, form the total length into a circle and center the hoop of wire at the center of the basketball. How much distance is there between the wire hoop and the basketball? Now do the same experiment with the earth. Stretch a wire around the equator of the earth, add exactly one foot of wire, and suppose the hoop of wire is centered at the center of the earth. How much distance is there between the wire and the earth?
3. In the sport of bobsledding there is a weight limit for the sled together with its riders. Recall that object fall in gravitational fields at the same rate independent of their masses. Why is the weight limit imposed?
4. Which number is larger π^e or e^π ? Prove that your answer is correct.
5. Prove the formula

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

is true for every positive integer n .

6. Evaluate the integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx.$$

7. Find the general solution of the differential equation

$$\frac{dx}{dt} = x(1 - x).$$

8. Prove that the value of the integral

$$\int_a^{a+2\pi} e^{\sin \theta} d\theta$$

does not depend on the choice of the real number a .

9. Suppose two radar stations are located in the plane at $A = (0, 0)$ and $B = (0, 1)$. Radar echos of an object in the right half-plane are received T_A seconds after transmission from A at A and T_B seconds after transmission from B at B . Where is the object?

10. Find an injective differentiable function that maps the open upper half of the plane to the open unit disk.
11. Find a polynomial P such that $|P(x) - e^x| < 10^{-2}$ whenever $0 \leq x \leq 1$.
12. Consider the set of all $n \times n$ matrices $\mathcal{M}(n)$ and note that these may be identified with points in \mathbf{R}^{n^2} (just string out the components in one long row). Define the commutator of two matrices by

$$[A, B] := AB - BA.$$

Fix a matrix $A \in \mathcal{M}(n)$ and define the map $F : \mathcal{M}(n) \rightarrow \mathcal{M}(n)$ by

$$F(B) = [A, B].$$

Compute the derivative of F .

13. (a) Prove that every continuous function defined on the unit sphere \mathbf{R}^3 has a maximum value. (b) Find the maximum value of the function $f : \mathbf{R}^3 \rightarrow \mathbf{R}$, defined by $f(x, y, z) = 1 + xyz$, restricted to the unit sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$.
14. Suppose that $f : \mathbf{R} \rightarrow \mathbf{R}$ and $g : \mathbf{R} \rightarrow \mathbf{R}$ are continuously differentiable functions, $r \in \mathbf{R}$, $f(r) = 0$, and $f'(r) \neq 0$. Show that if $|\epsilon|$ is sufficiently small, then the function $f + \epsilon g$ has a real root.
15. Find a choice of a, b, c , and d so that the function given by

$$f(t) = (a + bt)e^{2t} + (c + dt)e^{3t}$$

has the maximum possible number of zeros.

16. Find the general solution of the differential equation

$$(\tan x + m \sin y) \frac{dy}{dx} = (\sin y - m \tan x) \cos y$$

where m is a constant. (Hint: This problem can be solved by guessing. There is a general method that will lead to a solution, but this method is not taught in elementary courses of differential equations. Maybe there is an easy method?)

17. Compute the line integral

$$\int_C e^x \cos y \, dx - e^x \sin y \, dy$$

where C is the circle $(x - 14)^2 + (y + 5)^2 = 12$.

18. Find a numerical approximation of the real root of the polynomial $x^3 + 10x^2 + 3x - 5 = 0$ that is correct to 10 decimal places.
19. Prove that all of the eigenvalues of the real matrix

$$A = \begin{pmatrix} a & \alpha & \beta \\ \alpha & b & \gamma \\ \beta & \gamma & c \end{pmatrix}$$

are real numbers.

20. (a) Determine the escape velocity from the earth. (b) Suppose instead of the usual inverse square law of universal gravitation, the law is of the form $A/r^2 - B/r^4$, where $A = GMm$, $B = \epsilon GMm$, and ϵ is some small positive number. What is the escape velocity in this new gravitational field?
21. The destructive seismic waves (called S-waves) have frequencies of approximately 0.05 cycles per second. Suppose you wish to design a seismometer as a simple spring and a mass (hung from the end of the spring) of one kilogram. Assume that damping in the spring-mass system is viscous damping of 0.001 kilograms/second times the velocity of the displacement of the mass. You have a choice of springs with different spring constants. What is the ideal spring constant (the one that gives the largest amplitude response to a seismic wave)?

22.

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{3 \cdot 4 \cdot 5} + \frac{1}{5 \cdot 6 \cdot 7} + \cdots = ?$$

23. Let Ω be a bounded region in \mathbf{R}^n with the usual coordinates $x = (x_1, x_2, \dots, x_n)$ and define, for each $i \in \{1, 2, \dots, n\}$,

$$m_i = \frac{\int_{\Omega} x_i dx}{\int_{\Omega} dx}.$$

The point $m = (m_1, m_2, \dots, m_n)$ is called the centroid of Ω . Let $g : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be given by $g(x) = b + Ax$ where b is a vector in \mathbf{R}^n and A is an invertible linear transformation of \mathbf{R}^n . Prove that $g(m)$ is the centroid of $g(\Omega)$.

24. Let B denote the close unit ball in \mathbf{R}^3 . Find the value of $\int_B \operatorname{div} X dx dy dz$ for the vector field X on \mathbf{R}^3 given by

$$X(x, y, z) = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} - 3xyz \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

25. Show that the sequence $\{f_n\}_{n=1}^{\infty}$ of functions defined on the interval $[0, 1]$ by

$$f_n(x) = \int_0^x \sin(e^{nt} - 1) dt$$

has a uniformly convergent subsequence.

26. Consider the matrix

$$A = \begin{pmatrix} 1 & 6 & 12 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Find a 3×3 -matrix B such that BAB^{-1} is a diagonal matrix.

27. Consider a cup of coffee and a cup of tea. Take exactly one teaspoon of coffee and stir it into the tea. Now take exactly one teaspoon of the mixture of tea and coffee and pour it into the coffee. Is there more coffee in the tea cup, more tea in the coffee cup, or are the alien amounts the same?

28. (a) Consider the closed unit interval on the x -axis of the Cartesian plane. Treat the interval as an infinitely thin wire. Bend it, stretch it, but don't break it and then put it back in the plane so that it fits in the semi-infinite strip $\{(x, y) : 0 \leq x \leq 1 \text{ and } y \geq 0\}$. Prove that there is at least one point on the bent wire such that it and its original position in the interval lie on the same vertical line. (b) Show by constructing an example, that the result is not true if broken wires are allowed.
29. Prove that there is a unique continuous function $f : [0, 1] \rightarrow \mathbf{R}$ such that

$$f(t) = 1 + \int_0^t \frac{1}{2} f(s/2) ds.$$

30. Prove that there is no diffeomorphism (continuously differentiable with a continuously differentiable inverse) mapping the sphere to the torus.

Round Two:

1. Prove that there are an infinite number of prime numbers.
2. Prove the theorem of Pythagoras: The sums of the squares of the lengths of the sides of a right triangle is the square of the length of its hypotenuse.
3. Describe the set of points in space equidistant from two skew (that is non intersecting) lines.
4. Prove that the base angles of an isosceles triangle are equal.
5. Prove the quadratic formula.
6. Find the formulas for the solutions of cubic and quartic polynomial equations. (Note: You are allowed to look this up in a book. You should write out all the details of the derivation.) Is there a formula for the solution of a general quintic polynomial?
7. Let $\langle u, v \rangle$ denote the inner product of the vectors u and v in \mathbf{R}^n and define $O(n)$ to be the set of all linear transformations A of \mathbf{R}^n such that

$$\langle Au, Av \rangle = \langle u, v \rangle.$$

- (a) Prove that $O(n)$ is a group with respect to composition of linear transformations. (Note: The set $O(n)$ is called the orthogonal group with respect to the inner product.) (b) Specify two non trivial subgroups of $O(n)$. (c) Prove that all the eigenvalues of a transformation A in $O(n)$ lie on the unit circle in the complex plane. (d) Find a representation of all the orthogonal linear transformations of \mathbf{R}^2 and describe how the plane is transformed by an orthogonal transformation. (e) Find a representation of all the orthogonal linear transformations of \mathbf{R}^3 .
8. Let $\langle u, v \rangle$ denote the inner product of the vectors u and v in \mathbf{R}^n . A linear transformations A of \mathbf{R}^n such that

$$\langle u, Av \rangle = \langle Au, v \rangle$$

is called symmetric. (a) Prove that all the eigenvalues of a symmetric matrix are real. (b) Prove that every symmetric matrix is diagonalizable; that is, there is some invertible matrix B such that BAB^{-1} is a diagonal matrix. Interesting partial results would be to prove this for 2×2 matrices, 3×3 matrices, etc.

9. (a) For real functions of a real variable, define the concepts “monotone function” and “jump discontinuity.” Prove that a monotone function has only jump discontinuities. (b) Prove that a monotone function has at most a countable number of discontinuities.
10. (a) Prove that the function $f : \mathbf{R}^3 \setminus \{0\} \rightarrow \mathbf{R}$ given by $(x, y, z) \mapsto (x^2 + y^2 + z^2)^{-1/2}$ is harmonic (that is, the Laplacian of f is zero). (b) Show that the flux of ∇f through every surface in $\mathbf{R}^3 \setminus \{0\}$ whose boundary is the unit circle in the (x, y) -plane vanishes. (c) Is the same fact true for every harmonic function?
11. A home owner has a mortgage of \$100,000 at 5% interest (annual rate). If the monthly payment is \$1,500 per month, how many years will it take to pay off the loan.
12. You drop a coin off a bridge and notice that it takes three seconds to hit the river below. How high is the bridge?
13. A body falls according to the law

$$\frac{dv}{dt} = 4 - 2v^2, \quad v(0) = v_0,$$

where v denotes the velocity of the body. What is its terminal velocity?

14. Prove that

$$\sum_{j=0}^n \binom{n}{j} 2^j = 3^n,$$

where $\binom{n}{j} = n!/(j!(n-j)!)$.

15. Let $\langle u, v \rangle$ denote the inner product of the vectors u and v in \mathbf{R}^n , and let A be a linear transformation of \mathbf{R}^n . (a) Determine the derivative of the function $f : \mathbf{R}^n \rightarrow \mathbf{R}$ given by

$$f(u) = \langle u, Au \rangle.$$

(b) Prove that the critical points of the function f restricted to the unit sphere in \mathbf{R}^n are eigenvectors of the symmetric matrix A .

16. Let A be a linear transformation of \mathbf{R}^n . (a) Prove that the series

$$I + \sum_{j=1}^{\infty} \frac{1}{j!} A^j$$

converges to a linear transformation of \mathbf{R}^n . This transformation is usually denoted by e^A and called the exponential of the transformation A . Prove that e^A is an invertible linear transformation of \mathbf{R}^n and find its inverse.

17. A linear transformation is represented by the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

in the usual basis of \mathbf{R}^2 . What is its matrix representation in the basis

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

18. Find the minimum distance from the point $(5, 1)$ to graph of the relation $y = x^2$ in the plane.
19. A manufacturer is making cans (right circular cylinders with tops and bottoms) that must have volume V . Determine the optimum dimensions for the cans that will minimize the cost of material, which is the same for all parts.
20. A real function f of a real variable is infinitely differentiable. Suppose that f is decreasing and $\lim_{x \rightarrow \infty} f(x) = 0$. Does $\lim_{x \rightarrow \infty} f'(x) = 0$?
21. Suppose that the displacement of a spring is modeled by the differential equations

$$\ddot{x} + \lambda \dot{x} + kx = 0,$$

where λ and k are positive constants and the dots denote derivatives with respect to time. (a) Show that

$$\lim_{t \rightarrow \infty} x(t) = 0$$

for every solution $t \mapsto x(t)$ of the differential equation. (b) A model for the displacement of a periodically forced spring is

$$\ddot{x} + \lambda \dot{x} + kx = A \sin \omega t,$$

where ω is a positive constant. Describe the behavior of the solutions as time approaches infinity.

22. Prove that

$$0 < \int_0^{\infty} \frac{1}{(1+x^2)^3} + x^2 e^{-x^2} dx < \frac{3}{2}.$$

23. Let f be a continuous function on some interval containing a . Determine the limit

$$\lim_{h \rightarrow 0} \frac{1}{2h} \int_{a-h}^{a+h} f(x) dx.$$

24. Prove that the function $f : (0, \infty) \rightarrow \mathbf{R}$ given by

$$f(x) = x - \int_0^x \frac{\sin t}{t} dt.$$

is positive.

25. Suppose that A is an $n \times n$ real matrix and b an $n \times 1$ matrix of real numbers. Prove that the system of linear equations

$$Ax = b$$

has a unique solution for every such b if and only if the columns of A are linearly independent.

1. Suppose that $f : [0, 1] \rightarrow [0, 1]$ is continuous and $f(x) > 0$ for every $x \in [0, 1]$. Prove that there is some number $\lambda > 0$ such that $f(x) \geq \lambda$ for every $x \in [0, 1]$.
2. A set S (in Euclidean Space) is called disconnected if there are two open sets U and V such that $U \cap V = \phi$, $S \cap U \neq \phi$, $S \cap V \neq \phi$, and $S \subseteq U \cup V$. Assume that the real numbers are complete. Prove that the interval $[0, 1]$ is connected (i.e. not disconnected). Generalize your result to all intervals.
3. Consider the vector space of all polynomials \mathcal{P}_n of degree n and consider the usual derivative as a mapping on the set \mathcal{P}_n . (a) Prove that this is a linear transformation. (b) Find its kernel. (c) Consider the monomials $\{1, t, t^2, \dots, t^n\}$. Prove that these are a basis for \mathcal{P}_n and find the matrix of the transformation with respect to this basis. (d) The first four Chebyshev polynomials are

$$1, \quad t, \quad 2t^2 - 1, \quad 4t^3 - 3t.$$

Prove that these are a basis for \mathcal{P}_3 . Find the matrix of the transformation with respect to this basis. Also, find the linear map that gives the change of basis from the first basis to the Chebyshev basis. (Note: The Chebyshev polynomials can be defined as follows: $T_0 = 1$, $T_1 = t$, and

$$T_{n+1} = 2tT_n - T_{n-1}.$$

They are important because they satisfy an orthogonality property

$$\int_{-1}^1 T_n(t)T_m(t) \frac{1}{\sqrt{1-t^2}} dt$$

vanishes if and only if $n = m$.)

4. Prove that the Chebyshev differential equation

$$(1 - t^2)\ddot{x} - t\dot{x} + n^2x = 0$$

has a polynomial solution. Find the polynomial for $n = 0, 1, 2, 3$.

5. Prove that $\sqrt{2}$ is not a rational number.
6. Prove that a (square) matrix and its transpose have the same eigenvalues.
7. Prove that every eigenvalue of a symmetric matrix is real.
8. Prove that a symmetric matrix is diagonalizable.
9. (a) Find a solution of the partial differential equation $u_t = u_{xx}$ with $u(0, t) = 0$, $u(\pi, t) = 0$ and $u(0, x) = \sin x$ on the interval $0 < x < \pi$ and $t > 0$. (b) Solve the same problem only with $u(0, x) = x(\pi - x)$.
10. Prove Schwarz's inequality: If v and w are vectors in \mathbf{R}^n then the absolute value of their inner product is less than or equal to the product of their lengths; in symbols

$$|v \cdot w| \leq |v||w|.$$

1. Among all right circular cylinders that can be inscribed into a (fixed) sphere, find the dimensions of the one with greatest volume.
2. Consider the line integral

$$I = \int_C \frac{x}{x^2 + y^2} dx + \frac{y}{x^2 + y^2} dy.$$

- (a) Compute the value of this integral over the square with corners $(\pm 1, \pm 1)$ and the circle $x^2 + y^2 = 4$, where each figure is traversed in the counterclockwise direction. (b) Is there anything interesting going on here?